

SOME MODULI SPACES OF BRIDGELAND'S STABILITY CONDITIONS

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ABSTRACT. We shall study some moduli spaces of Bridgeland's semi-stable objects on abelian surfaces and K3 surfaces with Picard number 1. Under some conditions, we show that the moduli spaces are isomorphic to the moduli spaces of Gieseker semi-stable sheaves. We also study the ample cone of the moduli spaces.

0. INTRODUCTION.

Let X be an abelian surface or a K3 surface over a field \mathbf{k} . Denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on X , by $\mathbf{D}(X)$ the bounded derived category of $\mathrm{Coh}(X)$ and by $K(X)$ the Grothendieck group of $\mathbf{D}(X)$.

Let us fix an ample divisor H on X . For $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $\omega \in \mathbb{Q}_{>0}H$, Bridgeland [4] constructed a stability condition $\sigma_{\beta,\omega} = (\mathfrak{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ on $\mathbf{D}(X)$. Here $\mathfrak{A}_{(\beta,\omega)}$ is a tilting of $\mathrm{Coh}(X)$, and $Z_{(\beta,\omega)} : K(X) \rightarrow \mathbb{C}$ is a group homomorphism called the stability function. In terms of the Mukai lattice $(H^*(X, \mathbb{Z})_{\mathrm{alg}}, \langle \cdot, \cdot \rangle)$, $Z_{(\beta,\omega)}$ is given by

$$Z_{(\beta,\omega)}(E) = \langle e^{\beta + \sqrt{-1}\omega}, v(E) \rangle, \quad E \in K(X).$$

Here $v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}_X}$ is the Mukai vector of E . Hereafter for an object $E \in \mathbf{D}(X)$, we abbreviate write $Z_{(\beta,\omega)}(E) := Z_{(\beta,\omega)}([E])$, where $[E]$ is the class of E in $K(X)$. Let $\phi_{(\beta,\omega)} : \mathfrak{A}_{(\beta,\omega)} \setminus \{0\} \rightarrow (0, 1]$ be the phase function, which is defined to be $Z_{(\beta,\omega)}(E) = |Z_{(\beta,\omega)}(E)|e^{\pi\sqrt{-1}\phi_{(\beta,\omega)}(E)}$ for $0 \neq E \in \mathfrak{A}_{(\beta,\omega)}$.

Let w_1 be a primitive isotropic Mukai vector of an object in $\mathfrak{A}_{(\beta,\omega)}$. In this note, we shall study semi-stable objects E with respect to (β, ω) such that $\phi_{(\beta,\omega)}(E) = \phi_{(\beta,\omega)}(w_1)$. Assume that there is a coarse moduli scheme $M_{(\beta,\omega)}(w_1)$ of stable objects and $M_{(\beta,\omega)}(w_1)$ is projective. In the case where X is an abelian surface, [9] implies that this assumption is satisfied for any pair (β, ω) . Indeed $M_{(\beta,\omega)}(w_1)$ is the moduli space of semi-homogeneous sheaves (up to shift).

We set $X_1 := M_{(\beta,\omega)}(w_1)$. Let \mathbf{E} be a universal family as a complex of twisted sheaves on $X \times X_1$. Let $\Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]} : \mathbf{D}(X) \rightarrow \mathbf{D}^\alpha(X_1)$ be a twisted Fourier-Mukai transform by \mathbf{E} , where $(\cdot)^\vee := \mathbf{R}\mathcal{H}om(\cdot, \mathcal{O})$ is the derived dual, and α is a representative of a suitable Brauer class $[\alpha] \in H_{\mathrm{et}}^2(X_1, \mathcal{O}_{X_1}^\times)$. For simplicity, we set $\Phi := \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}$ and $\hat{\Phi} := \Phi_{X_1 \rightarrow X}^{\mathbf{E}[1]}$.

For $v \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ and $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$, we set

$$(0.1) \quad r_\beta(v) := -\langle v, \varrho_X \rangle, \quad a_\beta(v) := -\langle v, e^\beta \rangle, \quad d_\beta(v) := \frac{\langle v, H + (H, \beta)\varrho_X \rangle}{(H^2)}.$$

Then

$$(0.2) \quad v = r_\beta(v)e^\beta + a_\beta(v)\varrho_X + (d_\beta(v)H + D_\beta(v)) + (d_\beta(v)H + D_\beta(v), \beta)\varrho_X, \quad D_\beta(v) \in H^\perp \cap \mathrm{NS}(X)_{\mathbb{Q}}.$$

For a pair (β, ω) , $\mathcal{M}_{(\beta,\omega)}(v)$ denotes the moduli stack of $\sigma_{(\beta,\omega)}$ -semi-stable objects E with $v(E) = v$. Then we have the following result.

Theorem 0.0.1 (Theorem 3.3.3). *Let X be an abelian surface, or a K3 surface with $\mathrm{NS}(X) = \mathbb{Z}H$. Assume the following conditions:*

- (1) *There is a smooth projective surface X_1 which is the moduli space $M_{(\beta,\omega)}(w_1)$ of stable objects E with $v(E) = w_1$.*
- (2) *(β, ω) satisfies*

$$\langle d_\beta(v)w_1 - d_\beta(w_1)v, e^{\beta + \sqrt{-1}\omega} \rangle = 0.$$

- (3) *(β, ω) does not belong to any wall for v , or w_1 defines a wall W_{w_1} for v and (β, ω) belongs to exactly one wall W_{w_1} .*

2010 *Mathematics Subject Classification.* 14D20.

The second author is supported by JSPS Fellowships for Young Scientists (No. 21-2241). The third author is supported by the Grant-in-aid for Scientific Research (No. 22340010), JSPS.

Then for a general (β', ω') in a neighborhood of (β, ω) such that $(\beta', H) = (\beta, H)$, there is an ample divisor H_1 on X_1 such that $\mathcal{M}_{(\beta', \omega')}(v)$ is isomorphic to the moduli stack $\mathcal{M}_{H_1}(u)^{ss}$ of Gieseker semi-stable (twisted) sheaves with the Mukai vector u , where $u = \Phi(v)$ or $u = \Phi(v)^\vee$. In particular, there is a coarse moduli scheme $M_{(\beta', \omega')}(v)$ which is isomorphic to the moduli scheme of Gieseker semi-stable sheaves $\overline{M}_{H_1}(u)$.

As a corollary of this theorem, we get the following.

Theorem 0.0.2 (Theorem 4.1.1). *Let X be an abelian surface or a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. Assume that (β, ω) is general with respect to v . Then there is a coarse moduli scheme $M_{(\beta, \omega)}(v)$ which is isomorphic to the projective scheme $\overline{M}_{\hat{H}}(\Phi(v))$, where \hat{H} is a natural ample class on X_1 associated to H .*

If X is an abelian surface, we can also study the ample cone of $M_{(\beta, \omega)}(v)$ by using this theorem (Corollary 4.3.3).

1. PRELIMINARIES

As in the introduction, let X be an abelian surface or a K3 surface over a field \mathfrak{k} , and fix an ample divisor H on X .

1.1. Notations for Mukai lattice. We set $A_{\text{alg}}^*(X) = \bigoplus_{i=0}^2 A_{\text{alg}}^i(X)$ to be the quotient of the cycle group of X by the algebraic equivalence. Then we have $A_{\text{alg}}^0(X) \cong \mathbb{Z}$, $A_{\text{alg}}^1(X) \cong \text{NS}(X)$ and $A_{\text{alg}}^2(X) \cong \mathbb{Z}$. We denote the fundamental class of $A_{\text{alg}}^2(X)$ by ϱ_X , and express an element $x \in A_{\text{alg}}^*(X)$ by $x = x_0 + x_1 + x_2 \varrho_X$ with $x_0 \in \mathbb{Z}$, $x_1 \in \text{NS}(X)$ and $x_2 \in \mathbb{Z}$. The lattice structure $\langle \cdot, \cdot \rangle$ of $A_{\text{alg}}^*(X)$ is given by

$$(1.1) \quad \langle x, y \rangle := (x_1, y_1) - (x_0 y_2 + x_2 y_0),$$

where $x = x_0 + x_1 + x_2 \varrho_X$ and $y = y_0 + y_1 + y_2 \varrho_X$. We will call $(A_{\text{alg}}^*(X), \langle \cdot, \cdot \rangle)$ the Mukai lattice for X . In the case of $\mathfrak{k} = \mathbb{C}$, this lattice is sometimes denoted by $H^*(X, \mathbb{Z})_{\text{alg}}$ in literature. In this paper, we will use the symbol $H^*(X, \mathbb{Z})_{\text{alg}}$ even when \mathfrak{k} is arbitrary.

The Mukai vector $v(E) \in H^*(X, \mathbb{Z})_{\text{alg}}$ for $E \in \text{Coh}(X)$ is defined by

$$\begin{aligned} v(E) &:= \text{ch}(E) \sqrt{\text{td}_X} \\ &= \text{rk } E + c_1(E) + (\chi(E) - \varepsilon \text{rk } E) \varrho_X \in H^*(X, \mathbb{Z})_{\text{alg}} \end{aligned}$$

where $\varepsilon = 0, 1$ according as X is an abelian surface or a K3 surface. For an object E of $\mathbf{D}(X)$, $v(E)$ is defined by $\sum_k (-1)^k v(E^k)$, where $(E^k) = (\cdots \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots)$ is the bounded complex representing the object E .

For $\beta \in \text{NS}(X)_{\mathbb{Q}}$, we define the β -twisted semi-stability replacing the usual Hilbert polynomial $\chi(E(nH))$ by $\chi(E(-\beta + nH))$.

For a Mukai vector v , $\mathcal{M}_H^\beta(v)^{ss}$ denotes the moduli stack of β -twisted semi-stable sheaves E on X with $v(E) = v$. $\overline{M}_H^\beta(v)$ denotes the moduli scheme of S -equivalence classes of β -twisted semi-stable sheaves E on X with $v(E) = v$ and $M_H^\beta(v)$ denotes the open subscheme consisting of β -twisted stable sheaves. If $\beta = 0$, then we write $\overline{M}_H(v) := \overline{M}_H^\beta(v)$.

1.2. Stability conditions and wall/chamber structure. Let us recall the stability conditions given in [9, § 1]. For $E \in K(X)$ with (0.2), we have

$$(1.2) \quad \begin{aligned} Z_{(\beta, \omega)}(E) &= \langle e^{\beta + \sqrt{-1}\omega}, v(E) \rangle \\ &= -a_\beta(E) + \frac{(\omega^2)}{2} r_\beta(E) + d_\beta(E)(H, \omega) \sqrt{-1}. \end{aligned}$$

Then $\mathfrak{A}_{(\beta, \omega)}$ is the tilt of $\text{Coh}(X)$ with respect to a torsion pair $(\mathfrak{T}_{(\beta, \omega)}, \mathfrak{F}_{(\beta, \omega)})$ defined by

- (i) $\mathfrak{T}_{(\beta, \omega)}$ is generated by β -twisted stable sheaves with $Z_{(\beta, \omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$.
- (ii) $\mathfrak{F}_{(\beta, \omega)}$ is generated by β -twisted stable sheaves with $-Z_{(\beta, \omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$,

where $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ is the upper half plane.

For a chosen $\beta \in \text{NS}(X)_{\mathbb{Q}}$, let us write $b := (\beta, H)/(H^2) \in \mathbb{Q}$. Then $\beta = bH + \eta$ with $\eta \in H^\perp \cap \text{NS}(X)_{\mathbb{Q}}$. Now let us set

$$\mathfrak{H} := \{(\eta, \omega) \mid \eta \in \text{NS}(X)_{\mathbb{Q}}, (\eta, H) = 0, \omega \in \mathbb{Q}_{>0}H\}.$$

In [9, § 1.4], we showed that the category $\mathfrak{A}_{(bH+\eta, \omega)}$ changes only when (η, ω) moves across the wall for categories. Let us recall its definition.

Definition 1.2.1. Set

$$\mathfrak{R} := \{u \in A_{\text{alg}}^*(X) \mid u \in (H + (H, bH)\varrho_X)^\perp, \langle u^2 \rangle = -2\}.$$

- (1) For $u \in \mathfrak{R}$, we define a *wall* W_u for categories by

$$W_u := \{(\eta, \omega) \in \mathfrak{H}_{\mathbb{R}} \mid \text{rk } u \cdot (\omega^2) = -2\langle e^{bH+\eta}, u \rangle\},$$

where $\mathfrak{H}_{\mathbb{R}}$ is the enlarged parameter space defined by

$$\mathfrak{H}_{\mathbb{R}} := \{(\eta, \omega) \mid \eta \in \text{NS}(X)_{\mathbb{R}}, (\eta, H) = 0, \omega \in \mathbb{R}_{>0}H\}.$$

- (2) A connected component of $\mathfrak{H}_{\mathbb{R}} \setminus \cup_{u \in \mathfrak{R}} W_u$ is called a *chamber for categories*.

If X is an abelian surface or $(\omega^2) > 2$, then $\mathfrak{A}_{(bH+\eta, \omega)}$ does not depend on the choice of ω .

Now, the pair $\sigma_{(\beta, \omega)} = (\mathfrak{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ satisfies the requirement of stability conditions on $\mathbf{D}(X)$, as mentioned in the introduction (see [9, § 1.3] for the proof). In particular, the (semi-)stability of objects in $\mathfrak{A}_{(\beta, \omega)}$ with respect to $Z_{(\beta, \omega)}$ is well-defined.

Definition 1.2.2. $E \in \mathbf{D}(X)$ is called *semi-stable* of phase ϕ , if there is an integer n such that $E[-n]$ is a semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ with $\phi_{(\beta, \omega)}(E[-n]) = \phi - n$. If we want to emphasize the dependence on the stability condition, we say that E is $\sigma_{(\beta, \omega)}$ -*semi-stable*.

Definition 1.2.3. For a non-zero Mukai vector $v \in H^*(X, \mathbb{Z})_{\text{alg}}$, we define $Z_{(\beta, \omega)}(v) \in \mathbb{C}$ and $\phi_{(\beta, \omega)}(v) \in (0, 2]$ by

$$(1.3) \quad Z_{(\beta, \omega)}(v) := \langle e^{\beta + \sqrt{-1}\omega}, v \rangle = |Z_{(\beta, \omega)}(v)| e^{\pi \sqrt{-1} \phi_{(\beta, \omega)}(v)}.$$

Then

$$\phi_{(\beta, \omega)}(v(E)) = \phi_{(\beta, \omega)}(E)$$

for $0 \neq E \in \mathfrak{A}_{(\beta, \omega)} \cup \mathfrak{A}_{(\beta, \omega)}[1]$.

Definition 1.2.4. For a Mukai vector v , $\mathcal{M}_{(\beta, \omega)}(v)$ denotes the moduli stack of $\sigma_{(\beta, \omega)}$ -semi-stable objects E of $\mathfrak{A}_{(\beta, \omega)}$ with $v(E) = v$. $M_{(\beta, \omega)}(v)$ denotes the moduli scheme of the S -equivalence classes of $\sigma_{(\beta, \omega)}$ -semi-stable objects E of $\mathfrak{A}_{(\beta, \omega)}$ with $v(E) = v$, if it exists.

Next we recall the wall/chamber structure for *stabilities* (see [9, § 3.1] for details). Let us set the rational number

$$d_{\beta, \min} := \frac{1}{(H^2)} \min\{\deg(E(-\beta)) > 0 \mid E \in K(X)\} \in \frac{1}{d[(\beta, H)](H^2)} \mathbb{Z},$$

where $d[x]$ is the denominator of $x \in \mathbb{Q}$. Then $d_{\beta}(E) \in \mathbb{Z}d_{\beta, \min}$ for any $E \in K(X)$.

Definition 1.2.5. Let \mathcal{C} be a chamber for categories, that is, $\mathfrak{A}_{(bH+\eta, \omega)}$ is constant for $(\eta, \omega) \in \mathcal{C} \cap \mathfrak{H}$. For a Mukai vector v , let us set $r := r_{bH+\eta}(v)$, $d := d_{bH+\eta}(v)$ and $a := a_{bH+\eta}(v)$ using (0.1).

- (1) Let v_1 be a Mukai vector, and set $r_1 := r_{bH+\eta}(v_1)$, $d_1 := d_{bH+\eta}(v_1)$ and $a_1 := a_{bH+\eta}(v_1)$. For v_1 satisfying

- (a) $0 < d_1 < d$,
- (b) $\langle v_1^2 \rangle < (d_1/d) \langle v^2 \rangle + 2dd_1\varepsilon/d_{bH+\eta, \min}^2$,
- (c) $\langle v_1^2 \rangle \geq -2d_1^2\varepsilon/d_{bH+\eta, \min}^2$,

we define the *wall for stabilities of type v_1* as the set of

$$W_{v_1} := \{(\eta, \omega) \in \mathfrak{H}_{\mathbb{R}} \mid (\omega^2)(dr_1 - d_1r) = 2(-d\langle e^{bH+\eta}, v_1 \rangle + d_1\langle e^{bH+\eta}, v \rangle)\}.$$

- (2) A *chamber for stabilities* is a connected component of $\mathcal{C} \setminus \cup_{v_1} W_{v_1}$.

If it is necessary to emphasize the dependence on v , then we call W_{v_1} by the *wall for v* . By [9, Lemma 3.1.6], if (η, ω) and (η', ω') belong to the same chamber, then $\mathcal{M}_{(bH+\eta, \omega)}(v) = \mathcal{M}_{(bH+\eta', \omega')}(v)$. As we explained in [9], the above conditions (a),(b),(c) are necessary numerical conditions for the walls of stability conditions. Thus there may exist W_{v_1} such that the stability condition does not change by crossing W_{v_1} . For an abelian surface, we will give a necessary and sufficient condition of the wall where the stability condition does change in §4.2.

For the later discussions, we prepare

Definition 1.2.6. For a complex $F \in \mathbf{D}(X)$, let ${}^{\beta}H^p(F) \in \mathfrak{A}_{(\beta, \omega)}$ denote the p -th cohomology group of F with respect to the t -structure of $\mathbf{D}(X)$ associated to $\mathfrak{A}_{(\beta, \omega)}$.

By [3, Thm. 1.3.6], ${}^{\beta}H^p$ is a cohomological functor.

1.3. Some calculation of Mukai vector. As mentioned at (0.2) in the introduction, if $\beta \in \text{NS}(X)_{\mathbb{Q}}$ is chosen, then any $v \in H^*(X, \mathbb{Z})_{\text{alg}}$ can be expressed as

$$v = r_{\beta}(v)e^{\beta} + a_{\beta}(v)\varrho_X + (d_{\beta}(v)H + D_{\beta}(v)) + (d_{\beta}(v)H + D_{\beta}(v), \beta)\varrho_X, \quad D_{\beta}(v) \in H^{\perp} \cap \text{NS}(X)_{\mathbb{Q}}.$$

with $r_{\beta}(v), a_{\beta}(v), d_{\beta}(v)$ given by (0.1). The next lemma calculates the dependence of $a_{\beta}(v), d_{\beta}(v)$ on β , which will be repeatedly used in our discussion.

Lemma 1.3.1. *For $v \in H^*(X, \mathbb{Z})_{\text{alg}}$ and $\beta, \gamma \in \text{NS}(X)_{\mathbb{Q}}$,*

$$\begin{aligned} d_{\beta}(v)H + D_{\beta}(v) &= r_{\gamma}(v)(\gamma - \beta) + d_{\gamma}(v)H + D_{\gamma}(v), \\ a_{\beta}(v) &= a_{\gamma}(v) + (d_{\gamma}(v)H + D_{\gamma}(v), \gamma - \beta) + \frac{r_{\gamma}(v)}{2}((\beta - \gamma)^2). \end{aligned}$$

In particular,

$$d_{\beta}(v) = d_{\gamma}(v) + r_{\gamma}(v) \frac{\deg(\gamma - \beta)}{(H^2)}.$$

Proof. We note that

$$\begin{aligned} e^{\gamma} &= e^{\beta} e^{\gamma - \beta} = e^{\beta} + (\gamma - \beta + (\gamma - \beta, \beta)\varrho_X) + \frac{(\gamma - \beta)^2}{2} \varrho_X, \\ (d_{\gamma}(v)H + D_{\gamma}(v), \gamma) &= (d_{\gamma}(v)H + D_{\gamma}(v), \beta) + (d_{\gamma}(v)H + D_{\gamma}(v), \gamma - \beta). \end{aligned}$$

Hence we get

$$\begin{aligned} v &= r_{\gamma}(v)e^{\gamma} + a_{\gamma}(v)\varrho_X + (d_{\gamma}(v)H + D_{\gamma}(v) + (d_{\gamma}(v)H + D_{\gamma}(v), \gamma)\varrho_X) \\ &= r_{\gamma}(v)e^{\beta} + \left(a_{\gamma}(v) + (d_{\gamma}(v)H + D_{\gamma}(v), \gamma - \beta) + \frac{r_{\gamma}(v)}{2}((\beta - \gamma)^2) \right) \varrho_X \\ &\quad + (r_{\gamma}(v)(\gamma - \beta) + d_{\gamma}(v)H + D_{\gamma}(v) + (r_{\gamma}(v)(\gamma - \beta) + d_{\gamma}(v)H + D_{\gamma}(v), \beta)\varrho_X), \end{aligned}$$

which implies the claims. \square

1.4. The homological correspondence. For $\gamma \in \text{NS}(X)_{\mathbb{Q}}$, let

$$\begin{aligned} (1.4) \quad w_1 &:= r_1 e^{\gamma} \\ &= r_1 \left(e^{\beta} + \frac{(\beta - \gamma)^2}{2} \varrho_X + (\gamma - \beta + (\gamma - \beta, \beta)\varrho_X) \right) \in H^*(X, \mathbb{Z})_{\text{alg}} \end{aligned}$$

be a primitive isotropic Mukai vector such that $r_1(\gamma - \beta, H) > 0$.

Assume that there is a coarse moduli scheme $X_1 := M_{(\beta, \omega)}(w_1)$ of stable objects and X_1 is projective. Let \mathbf{E} be a universal object on $X \times X_1$ as a complex of twisted sheaves. We have $v(\mathbf{E}|_{X \times \{x_1\}}) = w_1$, $x_1 \in X_1$. We set $v(\mathbf{E}|_{\{x\} \times X_1}) = r_1 e^{\gamma'}$, $x \in X$.

Definition 1.4.1. For $C \in \text{NS}(X)_{\mathbb{Q}}$, we define $\widehat{C} \in \text{NS}(X_1)_{\mathbb{Q}}$ by

$$(1.5) \quad \Phi(C + (C, \gamma)\varrho_X) = \begin{cases} \widehat{C} + (\widehat{C}, \gamma')\varrho_{X_1}, & r_1 > 0, \\ -(\widehat{C} + (\widehat{C}, \gamma')\varrho_{X_1}), & r_1 < 0. \end{cases}$$

Lemma 1.4.2. (1) *If C belongs to the positive cone, then \widehat{C} belongs to the positive cone.*

(2) *Let H be an ample divisor on X . Assume that one of the following conditions holds:*

- (a) *X is an abelian surface,*
- (b) *$\text{NS}(X) = \mathbb{Z}H$,*
- (c) *$M_{(\beta, \omega)}(w_1)$ is the moduli of μ -stable vector bundles, that is,*

$$M_{(\beta, \omega)}(w_1) = \begin{cases} M_H(w_1), & \text{rk } w_1 > 0, \\ M_H(-w_1), & \text{rk } w_1 < 0. \end{cases}$$

Then \widehat{H} is ample. Thus we have the following.

1. *If $r_1 > 0$, then*

$$\begin{aligned} \Phi(e^{\gamma}) &= -\frac{1}{r_1} \varrho_{X_1}, \quad \Phi(\varrho_X) = -r_1 e^{\gamma'}, \\ \Phi(dH + D + (dH + D, \gamma)\varrho_X) &= d\widehat{H} + \widehat{D} + (d\widehat{H} + \widehat{D}, \gamma')\varrho_{X_1}, \end{aligned}$$

where $D \in \text{NS}(X)_{\mathbb{Q}} \cap H^{\perp}$.

2. If $r_1 < 0$, then

$$\begin{aligned}\Phi(e^\gamma) &= -\frac{1}{r_1}\varrho_{X_1}, \quad \Phi(\varrho_X) = -r_1 e^{\gamma'}, \\ \Phi(dH + D + (dH + D, \gamma)\varrho_X) &= -\left(d\hat{H} + \hat{D} + (d\hat{H} + \hat{D}, \gamma')\varrho_{X_1}\right),\end{aligned}$$

where $D \in \text{NS}(X)_{\mathbb{Q}} \cap H^{\perp}$.

Proof. (1) If $r_1 > 0$, then

$$\Phi(e^\gamma) = -\frac{1}{r_1}\varrho_{X_1}, \quad \Phi(\varrho_X) = -r_1e^{\gamma'}, \quad \Phi(C + (C, \gamma)\varrho_X) = \widehat{C} + (\widehat{C}, \gamma')\varrho_{X_1}.$$

If $r_1 < 0$, then

$$\Phi(e^\gamma) = -\frac{1}{r_1}\varrho_{X_1}, \quad \Phi(\varrho_X) = -r_1 e^{\gamma'}, \quad \Phi(C + (C, \gamma)\varrho_X) = -(\widehat{C} + (\widehat{C}, \gamma')\varrho_{X_1}).$$

Since Φ preserves the orientation of the Mukai lattice ([6] or Proposition 3.4.3 for a K3 surface, and [12] for an abelian surface), if C belongs to the positive cone, then \widehat{C} also belongs to the positive cone.

(2) Assume that (a) or (b) holds. Then $\widehat{C} \in \text{NS}(X_1)_{\mathbb{Q}}$ is ample if and only if \widehat{C} belongs to the positive cone. Since H is ample, (1) implies the claim. If (c) holds, then it is known that \widehat{H} is ample by the construction of the moduli space. \square

1.5. A lemma on angles of stability functions. In this paper, we often compare the phases $\phi_{(\beta, \omega)}(E)$ and $\phi_{(\beta, \omega)}(F)$ of two objects $E, F \in \mathbf{D}(X)$. For that, it is convenient to use the next function, which was introduced in [9, § 1.3].

Definition 1.5.1. For $E, E' \in K(X)$, we set

$$\Sigma_{(\beta, \omega)}(E', E) := \det \begin{pmatrix} \operatorname{Re} Z_{(\beta, \omega)}(E') & \operatorname{Re} Z_{(\beta, \omega)}(E) \\ \operatorname{Im} Z_{(\beta, \omega)}(E') & \operatorname{Im} Z_{(\beta, \omega)}(E) \end{pmatrix}.$$

We also set $\Sigma_{(\beta, \omega)}(v', v) := \Sigma_{(\beta, \omega)}(E', E)$ for $v(E) = v, v(E') = v'$.

Then we have $\Sigma_{(\beta, \omega)}(E', E) \geq 0$ if and only if $\phi_{(\beta, \omega)}(E) - \phi_{(\beta, \omega)}(E') \geq 0$ (see [9, Remark 1.3.5]).

Next, let us prepare some notations for the phase of the stability function. For $\phi \in \mathbb{R}$, $P(\phi)$ denotes the category of semi-stable objects $E \in \mathbf{D}(X)$ with $\phi_{(\beta, \omega)}(E) = \phi$.

By the Harder-Narasimhan property of the stability function $Z_{(\beta, \omega)}$, for any $0 \neq E \in \mathbf{D}(X)$ we have a collection of triangles

$$0 = E_0 \xrightarrow{\quad} E_1 \xrightarrow{\quad} E_2 \xrightarrow{\quad} \cdots \xrightarrow{\quad} E_{n-1} \xrightarrow{\quad} E_n = E$$

$\swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow$
 $[1] \quad A_1 \quad [1] \quad A_2 \quad [1] \quad A_n$

such that $A_i \in P(\phi_i)$ with $\phi_1 > \phi_2 > \dots > \phi_n$. Let us denote $\phi_{\max}(E) := \phi_1$ and $\phi_{\min}(E) := \phi_n$.

Now we want to state the main Lemma 1.5.2 in this subsection. Let us recall the notations in the introduction: $\Phi := \Phi_{X \rightarrow X_1}^{\mathbf{E}'[1]}$ and $\hat{\Phi} := \Phi_{X_1 \rightarrow X}^{\mathbf{E}[1]}$.

Lemma 1.5.2. *We set $\phi := \phi_{(\beta, \omega)}$. For a torsion free (twisted) sheaf E on X_1 , $F := \widehat{\Phi}(E)$ satisfies the following properties.*

- (1) $\text{Hom}(\mathbf{E}|_{X \times \{x_1\}}, F[k]) = 0$ for $k \neq 0, 1$ and $\text{Hom}(\mathbf{E}|_{X \times \{x_1\}}, F) = 0$ except finitely many points $x_1 \in X_1$.
- (2) $\phi(w_1) - 1 < \phi_{\min}(F) \leq \phi_{\max}(F) < \phi(w_1) + 1$.
- (3) Assume that $Z_{(\beta, \omega)}(F) \in \mathbb{R}Z_{(\beta, \omega)}(w_1)$, i.e., $\Sigma_{(\beta, \omega)}(F, w_1) = 0$. If F is not a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$, then there is an exact sequence of torsion free sheaves

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that

$$\phi(w_1) + 1 > \phi_{\max}(\hat{\Phi}(E_1)) \geq \phi_{\min}(\hat{\Phi}(E_1)) > \phi(w_1)$$

and

$$\phi(w_1) \geq \phi_{\max}(\widehat{\Phi}(E_2)) \geq \phi_{\min}(\widehat{\Phi}(E_2)) > \phi(w_1) - 1.$$

In particular, $\Sigma_{(\beta, \omega)}(w_1, \widehat{\Phi}(E_1)) > 0$ and $\Sigma_{(\beta, \omega)}(\widehat{\Phi}(E_2), w_1) > 0$.

Proof. (1) We note that

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F[k]) = \mathrm{Hom}(\Phi(\mathbf{E}_{|X \times \{x_1\}}), \Phi(F)[k]) = \mathrm{Hom}(\mathbf{E}_{x_1}[-1], E[k]).$$

Hence $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F[k]) = 0$ for $k \neq 0, 1$. Since E is torsion free, we also have $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F) = 0$ except for finitely many points $x_1 \in X_1$. Thus (1) holds.

(2) We first prove that $\phi_{\max}(F) < \phi(w_1) + 1$. Assume that there is an exact triangle

$$(1.6) \quad F_1 \rightarrow F \rightarrow F_2 \rightarrow F_1[1]$$

with $\phi_{\min}(F_1) \geq \phi(w_1) + 1$ and $\phi_{\max}(F_2) < \phi(w_1) + 1$.

Since

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = \mathrm{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}[2 - k])^\vee$$

and $\phi(\mathbf{E}_{|X \times \{x_1\}}[2 - k]) - \phi_{\min}(F_1) \leq 1 - k$, we have

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = 0 \quad \text{for } k \geq 2,$$

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[1]) = 0 \quad \text{except for finitely many } x_1 \in X_1.$$

We also have $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[k]) = 0$ for $k \leq -1$, since $\phi_{\max}(F_2[k]) - \phi(\mathbf{E}_{|X \times \{x_1\}}) < 1 + k \leq 0$. Then taking $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, \bullet)$ of (1.6) and using (1), we find that

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = 0 \quad \text{for } k \leq -1,$$

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1) = 0 \quad \text{except for finitely many } x_1 \in X_1.$$

By $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = 0$ for $k \neq 0, 1$ and all $x_1 \in X_1$, $\Phi(F_1)$ is represented by a complex $V_{-1} \rightarrow V_0$ of locally free sheaves. Since $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = 0$, $k = 0, 1$ except for finitely many $x_1 \in X_1$, $\Phi(F_1) = 0$. Thus $F_1 = 0$.

We next prove that $\phi_{\min}(F) > \phi(w_1) - 1$. Assume that there is an exact triangle

$$(1.7) \quad F_1 \rightarrow F \rightarrow F_2 \rightarrow F_1[1]$$

such that $\phi_{\max}(F_2) \leq \phi(w_1) - 1$ and $\phi_{\min}(F_1) > \phi(w_1) - 1$. Then

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = \mathrm{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}[2 - k])^\vee = 0$$

for $k \geq 3$. We also have

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[k]) = 0 \quad \text{for } k \leq 0,$$

$$\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[1]) = 0 \quad \text{except for finitely many } x_1 \in X_1.$$

Then by (1.7), $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[k]) = 0$ for $k \neq 1$, which implies that $\Phi(F_2)$ is locally free. On the other hand, $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[1]) = 0$ except for finitely many $x_1 \in X_1$. Therefore $F_2 = 0$.

(3) We first prove that $\phi_{\max}(F) > \phi(w_1)$ and $\phi_{\min}(F) < \phi(w_1)$. If $\phi_{\max}(F) \leq \phi(w_1)$, then since $Z_{(\beta, \omega)}(F) \in \mathbb{R}Z_{(\beta, \omega)}(w_1)$, we see that $\phi_{\min}(F) = \phi_{\max}(F) = \phi(w_1)$. Thus F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$, which is a contradiction. If $\phi_{\min}(F) \geq \phi(w_1)$, then we also get that $\phi_{\min}(F) = \phi_{\max}(F) = \phi(w_1)$, which contradict our assumption on F . Therefore the claims hold.

Then we have an exact triangle

$$(1.8) \quad F_1 \rightarrow F \rightarrow F_2 \rightarrow F_1[1]$$

such that $\phi_{\min}(F_1) > \phi(w_1)$ and $\phi_{\max}(F_2) \leq \phi(w_1)$. By (2), we have $\phi_{\max}(F_1) < \phi(w_1) + 1$ and $\phi_{\min}(F_2) > \phi(w_1) - 1$. Since $\phi_{\min}(F_1) > \phi(w_1)$ and $\phi_{\max}(F_2) \leq \phi(w_1)$, we have $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_1[k]) = 0$ for $k \geq 2$ and $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2[k]) = 0$ for $k < 0$. Moreover $\mathrm{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F_2) = 0$ except finitely many $x_1 \in X_1$. We set $E_i := \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}(F_i)$. Then E_i are torsion free sheaves fitting in an exact sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0.$$

Since $\widehat{\Phi}(E_i) = F_i$, $i = 1, 2$, we get the claim. \square

2. RELATION WITH μ -SEMI-STABILITY.

In this section, we fix the pair (β, ω) and set $\phi := \phi_{(\beta, \omega)}$. We shall study the relation of Bridgeland stability with μ -semi-stability. The main statement is given in Theorem 2.2.1. We shall freely use the notations $\Phi := \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}$ and $\widehat{\Phi} := \Phi_{X_1 \rightarrow X}^{\mathbf{E}[1]}$.

2.1. A polarization of X_1 . In this subsection, we introduce a \mathbb{Q} -divisor \widehat{L} on X_1 and show that it is ample under suitable assumptions.

Lemma 2.1.1. *For a $\sigma_{(\beta,\omega)}$ -semi-stable object F of $\mathfrak{A}_{(\beta,\omega)}$ with $\phi(F) = \phi(w_1)$, we have an exact sequence in $\mathfrak{A}_{(\beta,\omega)}$*

$$(2.1) \quad 0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

such that F_1 is a $\sigma_{(\beta,\omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta,\omega)}$ satisfying $\phi(F_1) = \phi(w_1)$ and $\text{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$, and F_2 is S -equivalent to $\oplus_i \mathbf{E}_{|X \times \{x_i\}}$, $x_i \in X_1$. Since $\text{Hom}(F_1, F_2) = 0$, (2.1) is uniquely determined by F .

Proof. For a non-zero morphism $\varphi : F \rightarrow \mathbf{E}_{|X \times \{x_1\}}$, $\phi(F) = \phi(\mathbf{E}_{|X \times \{x_1\}})$ implies that φ is surjective and $\ker \varphi$ is a $\sigma_{(\beta,\omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta,\omega)}$ with $\phi(\ker \varphi) = \phi(w_1)$. Apply this procedure successively, we finally obtain a subobject F_1 of F such that $\phi(F_1) = \phi(w_1)$ and $\text{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$. Then $F_2 := F/F_1$ is S -equivalent to $\oplus_i \mathbf{E}_{|X \times \{x_i\}}$, $x_i \in X_1$. \square

We set $\gamma - \beta := \lambda H + \nu$, $(\nu, H) = 0$. Then we see that

$$(2.2) \quad \begin{aligned} & (a_\beta(v)d_\beta(w_1) - a_\beta(w_1)d_\beta(v))(H^2) \\ &= r_1 a_\gamma(v) \deg(\gamma - \beta) + r_1(d_\gamma(v)H + D_\gamma(v), \gamma - \beta) \deg(\gamma - \beta) - r_1 d_\gamma(v)(H^2) \frac{((\gamma - \beta)^2)}{2} \\ &= r_1 a_\gamma(v) \lambda(H^2) + r_1 \lambda^2 \frac{(H^2)^2}{2} d_\gamma(v) + r_1(D_\gamma(v), \nu) \lambda(H^2) - r_1 d_\gamma(v)(H^2) \frac{(\nu^2)}{2} \end{aligned}$$

and

$$(2.3) \quad r_\beta(v)d_\beta(w_1) - r_\beta(w_1)d_\beta(v) = r_\gamma(v)d_\gamma(w_1) - r_\gamma(w_1)d_\gamma(v) = -r_\gamma(w_1)d_\gamma(v).$$

Then

$$(2.4) \quad \begin{aligned} \frac{\Sigma_{(\beta,\omega)}(v, w_1)}{(H, \omega)} &= (r_\beta(v)d_\beta(w_1) - r_\beta(w_1)d_\beta(v)) \frac{(\omega^2)}{2} - (a_\beta(v)d_\beta(w_1) - a_\beta(w_1)d_\beta(v)) \\ &= -\frac{r_1}{2} ((\omega^2) + \lambda^2(H^2) - (\nu^2)) d_\gamma(v) - r_1 \lambda(a_\gamma(v) + (D_\gamma(v), \nu)). \end{aligned}$$

Definition 2.1.2. For $\gamma - \beta := \lambda H + \nu$, $\nu \in H^\perp$, we set

$$L := \frac{(\omega^2) + \lambda^2(H^2) - (\nu^2)}{2(H^2)} H + \lambda \nu \in \text{NS}(X)_\mathbb{Q}.$$

By (2.4), we get the following lemma.

Lemma 2.1.3.

$$(2.5) \quad \frac{\Sigma_{(\beta,\omega)}(v, w_1)}{(H, \omega)} = -r_1(c_1(v) - (\text{rk } v)\gamma, L) - r_1 \lambda a_\gamma(v).$$

Let us study the properties of L . We note that

$$(2.6) \quad \begin{aligned} (L^2) &= \frac{1}{4(H^2)} \left[((\omega^2) + \lambda^2(H^2) - (\nu^2))^2 + 4\lambda^2(H^2)(\nu^2) \right] \\ &> \frac{1}{4(H^2)} (\lambda^2(H^2) + (\nu^2))^2 \geq 0. \end{aligned}$$

Since $(L, H) > 0$, L belongs to the positive cone.

Lemma 2.1.4. *Assume that one of the following conditions holds:*

- (i) X is an abelian surface.
- (ii) $\text{NS}(X) = \mathbb{Z}H$.
- (iii) $\nu = 0$.

Then L is an ample \mathbb{Q} -divisor.

Proof. Since L belongs to the positive cone, L is an ample divisor, if X is an abelian surface or $\text{NS}(X) = \mathbb{Z}H$. If $\nu = 0$, then $L \in \mathbb{Q}_{>0}H$. Thus L is also an ample divisor. \square

Remark 2.1.5. (1) Assume that X is an abelian surface. Then since $(\widehat{L}^2) = (L^2) > 0$ and $(\widehat{L}, \widehat{H}) = (L, H) > 0$, \widehat{L} is ample.

(2) Assume that X is a $K3$ surface and one of the conditions (b), (c) of Lemma 1.4.2 (2) holds. If $\nu = 0$, then $\widehat{L} \in \mathbb{Q}_{>0}\widehat{H}$ is also ample.

Lemma 2.1.6. For $E \in K(X_1)$, $Z_{(\beta, \omega)}(\widehat{\Phi}(E)) \in \mathbb{R}Z_{(\beta, \omega)}(w_1)$ if and only if

$$(2.7) \quad (c_1(E) - \text{rk } E\gamma', \widehat{L}) = \frac{\lambda}{|r_1|} \text{rk } E.$$

Proof. We set $F := \widehat{\Phi}(E)$. By Lemma 1.4.2, we have

$$c_1(\widehat{F(-\gamma)}) = \frac{r_1}{|r_1|} c_1(\Phi(F)(-\gamma')) = \frac{r_1}{|r_1|} c_1(E(-\gamma')).$$

By using Lemma 2.1.3, we see that

$$(2.8) \quad \begin{aligned} \frac{\Sigma_{(\beta, \omega)}(F, w_1)}{(H, \omega)} &= -r_1(c_1(F(-\gamma)), L) - r_1 \lambda a_\gamma(F) \\ &= -r_1(c_1(\widehat{F(-\gamma)}), \widehat{L}) + \lambda \text{rk } E \\ &= -|r_1|(c_1(E(-\gamma')), \widehat{L}) + \lambda \text{rk } E. \end{aligned}$$

Therefore (2.7) holds. \square

We also have the following as a consequence of (2.8).

Lemma 2.1.7. For $w \in H^*(X, \mathbb{Z})_{\text{alg}}$ with $\text{rk } \Phi(w) = -r_1 a_\gamma(w) > 0$,

$$\Sigma_{(\beta, \omega)}(w, w_1) \begin{matrix} > \\ < \end{matrix} 0 \iff \frac{(c_1(\Phi(w)(-\gamma')), \widehat{L})}{\text{rk } \Phi(w)} \begin{matrix} < \\ > \end{matrix} \frac{\lambda}{|r_1|}.$$

2.2. μ -semi-stability and $\sigma_{(\beta, \omega)}$ -semi-stability. From now on, we assume that \widehat{L} is ample.

Theorem 2.2.1. Assume that \widehat{L} is ample. Let F be an object of $\mathbf{D}(X)$ such that $Z_{(\beta, \omega)}(F) \in \mathbb{R}Z_{(\beta, \omega)}(w_1)$.

(1) F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ if and only if $\Phi(F)$ fits in an exact triangle

$$E_1 \rightarrow \Phi(F) \rightarrow E_2[-1] \rightarrow E_1[1],$$

where E_1 is a μ -semi-stable torsion free sheaf on X_1 and E_2 is a 0-dimensional sheaf on X_1 .

(2) Assume that F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$.

(a) $\Phi(E)$ is a torsion free sheaf on X_1 if and only if $\text{Hom}(F, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$.

(b) $\Phi(E)^\vee$ is a torsion free sheaf on X_1 if and only if $\text{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F) = 0$ for all $x_1 \in X_1$.

Remark 2.2.2. We take $\beta' \in \text{NS}(X_1)_{\mathbb{Q}}$ with $(\beta' - \gamma', \widehat{L}) = \frac{\lambda}{|r_1|}$ and $\omega' \in \mathbb{R}_{>0}\widehat{L}$ with $(\widehat{L}^2) > 2$. Let F be an object of $\mathbf{D}(X)$ in Theorem 2.2.1. Then Theorem 2.2.1 implies that F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ if and only if $\Phi(F)[1]$ is a $\sigma_{(\beta', \omega')}$ -semi-stable object of $\mathfrak{A}_{(\beta', \omega')}$ with $\phi_{(\beta', \omega')}(\Phi(F)[1]) = 1$. In §5.1, we give a more precise relation with Bridgeland stability on X_1 .

Corollary 2.2.3. Assume that w_1 does not define a wall for v .

(1) Every $\sigma_{(\beta, \omega)}$ -semi-stable object F with $v(F) = v$ satisfies

$$\text{Hom}(\mathbf{E}_{|X \times \{x_1\}}, F) = \text{Hom}(F, \mathbf{E}_{|X \times \{x_1\}}) = 0$$

for all x_1 . Hence $\Phi(F)$ is a locally free μ -semi-stable sheaf.

(2) Moreover if (β, ω) does not belong any wall for v , then $\Phi(F)$ is δ -twisted semi-stable for all δ .

Proof. (2) Let E be a μ -semi-stable sheaf with $v(E) = \Phi(v)$. For an exact sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that E_i are torsion free sheaves with the same slope, if $v(E_i) \notin \mathbb{Q}v(E)$, then $\widehat{\Phi}(E_i)$ defines a wall for v . Hence $v(E_i) \in \mathbb{Q}v(E)$. Then E is δ -twisted semi-stable for any δ . \square

We divide the proof of Theorem 2.2.1 into the proofs of Proposition 2.2.4 and Proposition 2.2.5 below.

Proposition 2.2.4. Assume that \widehat{L} is ample. Let E be a torsion free sheaf on X_1 with

$$(2.9) \quad \frac{(c_1(E) - \text{rk } E\gamma', \widehat{L})}{\text{rk } E} = \frac{\lambda}{|r_1|}.$$

If E is μ -semi-stable with respect to \widehat{L} , then $F := \widehat{\Phi}(E) \in \mathfrak{A}_{(\beta, \omega)}$ and F is $\sigma_{(\beta, \omega)}$ -semi-stable with $\phi(F) = \phi(w_1)$.

Proof. Let E be a torsion free sheaf on X_1 with (2.9) such that E is μ -semi-stable with respect to \widehat{L} . By Lemma 2.1.6,

$$(2.10) \quad Z_{(\beta, \omega)}(\widehat{\Phi}(E)) \in \mathbb{R}Z_{(\beta, \omega)}(w_1).$$

Assume that $F := \widehat{\Phi}(E)$ is not $\sigma_{(\beta, \omega)}$ -semi-stable. By Lemma 1.5.2 (3), we have an exact sequence of torsion free sheaves

$$(2.11) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that $\Sigma_{(\beta, \omega)}(w_1, F_1) > 0$ and $\Sigma_{(\beta, \omega)}(F_2, w_1) > 0$, where $F_1 = \widehat{\Phi}(E_1)$ and $F_2 = \widehat{\Phi}(E_2)$.

Applying Lemma 2.1.7 to $w = v(F_2)$ and $w = v(F)$, we get

$$(2.12) \quad \frac{(c_1(E_2(-\gamma')), \widehat{L})}{\text{rk } E_2} < \frac{\lambda}{|r_1|} = \frac{(c_1(E(-\gamma')), \widehat{L})}{\text{rk } E},$$

which implies that E is not μ -semi-stable with respect to \widehat{L} . Therefore F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$. Then by (2.10), $\phi(F) = \phi(w_1)$. \square

Proposition 2.2.5. *Let F be an object of $\mathfrak{A}_{(\beta, \omega)}$ such that $\phi(F) = \phi(w_1)$ and F is $\sigma_{(\beta, \omega)}$ -semi-stable. Then $\Phi(F)$ fits in an exact triangle*

$$E_1 \rightarrow \Phi(F) \rightarrow E_2[-1] \rightarrow E_1[1],$$

where E_1 is a μ -semi-stable torsion free sheaf on X_1 and E_2 is a 0-dimensional sheaf on X_1 . Moreover E_1 is $\Phi(F)$ is a μ -stable sheaf, if F is a $\sigma_{(\beta, \omega)}$ -stable object.

Proof. By Lemma 2.1.1, we have an exact sequence

$$(2.13) \quad 0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

in $\mathfrak{A}_{(\beta, \omega)}$, where F_1 is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ such that $\phi(F_1) = \phi(w_1)$ and $\text{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$, and F_2 is S -equivalent to $\oplus_i \mathbf{E}_{|X \times \{x_i\}}$, $x_i \in X_1$. Applying $\Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}$, we have an exact triangle

$$\Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}(F_1) \rightarrow \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}(F) \rightarrow \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[2]}(F_2)[-1] \rightarrow \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}(F_1)[1].$$

Obviously $E_2 := \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[2]}(F_2)$ is a 0-dimensional sheaf on X_1 .

We shall prove that $E_1 := \Phi_{X \rightarrow X_1}^{\mathbf{E}^\vee[1]}(F_1)$ is a μ -semi-stable sheaf on X_1 . Since $\text{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$, E_1 is a torsion free sheaf on X_1 . Assume that E_1 is not μ -semi-stable with respect to \widehat{L} . Then we have an exact sequence

$$(2.14) \quad 0 \rightarrow E'_1 \rightarrow E_1 \rightarrow E''_1 \rightarrow 0$$

such that E'_1 and E''_1 are torsion free sheaves. Then we have an exact triangle

$$\widehat{\Phi}(E'_1) \rightarrow \widehat{\Phi}(E_1) \rightarrow \widehat{\Phi}(E''_1) \rightarrow \widehat{\Phi}(E'_1)[1].$$

By Lemma 1.5.2, we get

$$(2.15) \quad \begin{aligned} \phi(w_1) - 1 &< \phi_{\min}(\widehat{\Phi}(E'_1)) \leq \phi_{\max}(\widehat{\Phi}(E'_1)) < \phi(w_1) + 1 \\ \phi(w_1) - 1 &< \phi_{\min}(\widehat{\Phi}(E''_1)) \leq \phi_{\max}(\widehat{\Phi}(E''_1)) < \phi(w_1) + 1. \end{aligned}$$

In particular, using ${}^\beta H^p$ in Definition 1.2.6, ${}^\beta H^p(\widehat{\Phi}(E'_1)) = {}^\beta H^p(\widehat{\Phi}(E''_1)) = 0$ except for $p = -1, 0, 1$. Since $\widehat{\Phi}(E_1) = F_1 \in \mathfrak{A}_{(\beta, \omega)}$, we have ${}^\beta H^{-1}(\widehat{\Phi}(E'_1)) = {}^\beta H^1(\widehat{\Phi}(E''_1)) = 0$ and an exact sequence

$$0 \rightarrow {}^\beta H^{-1}(\widehat{\Phi}(E''_1)) \rightarrow {}^\beta H^0(\widehat{\Phi}(E'_1)) \xrightarrow{\psi} F_1 \rightarrow {}^\beta H^0(\widehat{\Phi}(E''_1)) \rightarrow {}^\beta H^1(\widehat{\Phi}(E'_1)) \rightarrow 0$$

in $\mathfrak{A}_{(\beta, \omega)}$. Then $\phi({}^\beta H^{-1}(\widehat{\Phi}(E''_1))) < \phi(w_1)$. By the $\sigma_{(\beta, \omega)}$ -semi-stability of F_1 , $\phi(\text{im } \psi) \leq \phi(w_1)$. Hence

$$\phi({}^\beta H^0(\widehat{\Phi}(E'_1))) \leq \phi(w_1).$$

Since $0 \geq \phi({}^\beta H^1(\widehat{\Phi}(E'_1))[-1]) > \phi(w_1) - 1$, we have

$$\phi(w_1) - 1 < \phi(\widehat{\Phi}(E'_1)) \leq \phi(w_1).$$

Assume that $\phi(\widehat{\Phi}(E'_1)) < \phi(w_1)$. By Lemma 2.1.7, we have

$$\frac{(c_1(E'_1), \widehat{L})}{\text{rk } E'_1} < \frac{(c_1(E_1), \widehat{L})}{\text{rk } E_1}.$$

If $\phi(\widehat{\Phi}(E'_1)) = \phi(w_1)$, then we have ${}^\beta H^{-1}(\widehat{\Phi}(E''_1)) = {}^\beta H^1(\widehat{\Phi}(E''_1)) = 0$. In this case, we have

$$(2.16) \quad \frac{(c_1(E'_1), \widehat{L})}{\text{rk } E'_1} = \frac{(c_1(E_1), \widehat{L})}{\text{rk } E_1}.$$

Hence E_1 is μ -semi-stable with respect to \widehat{L} . Moreover if F is $\sigma_{(\beta,\omega)}$ -stable, then $\Phi(F)$ is a μ -stable torsion free sheaf on X_1 . \square

3. RELATION WITH GIESEKER SEMI-STABILITY.

We shall study the relation of Bridgeland stability with Gieseker semi-stability by refining the arguments in the last subsection. After we prepare some calculations on the properly semi-stable objects in § 3.1, we introduce the adjacent chambers \mathcal{C}_\pm to a wall for stabilities in § 3.2. The wall crossing behavior for these chambers is studied in § 3.2 and § 3.3, and the main theorem (Theorem 3.3.3) will be obtained. In the final § 3.4, we study the assumption in Theorem 3.3.3 for $K3$ surfaces.

3.1. Properly semi-stable objects.

Lemma 3.1.1. *Let $v \in H^*(X, \mathbb{Z})_{\text{alg}}$ be a Mukai vector with $d_\beta(v) > 0$. Assume that (β, ω) belongs to exactly one wall W_{w_1} for v . For a subobject F_1 of F with $v(F) = v$ and $\phi_{(\beta,\omega)}(F_1) = \phi_{(\beta,\omega)}(F)$,*

$$\frac{c_1(F_1(-\gamma))}{a_\gamma(F_1)} = \frac{c_1(F(-\gamma))}{a_\gamma(F)}.$$

Proof. We set

$$\begin{aligned} v &= v(F) = r_\gamma(v)e^\gamma + a_\gamma(v)\varrho_X + (\xi + (\xi, \gamma)\varrho_X), \\ &= e^\gamma(r_\gamma(v) + a_\gamma(v)\varrho_X + \xi) \\ (3.1) \quad v_1 &= v(F_1) = r_\gamma(v_1)e^\gamma + a_\gamma(v_1)\varrho_X + (\xi_1 + (\xi_1, \gamma)\varrho_X) \\ &= e^\gamma(r_\gamma(v_1) + a_\gamma(v_1)\varrho_X + \xi_1). \end{aligned}$$

Since $d_\beta(v_1)v - d_\beta(v)v_1$ and $d_\beta(w_1)v - d_\beta(v)w_1$ define the same wall, they are linearly dependent.

We note that

$$\begin{aligned} (3.2) \quad & d_\beta(v_1)v - d_\beta(v)v_1 \\ &= e^\gamma[(d_\beta(v_1)r_\gamma(v) - d_\beta(v)r_\gamma(v_1)) + (d_\beta(v_1)a_\gamma(v) - d_\beta(v)a_\gamma(v_1))\varrho_X + (d_\beta(v_1)\xi - d_\beta(v)\xi_1)] \end{aligned}$$

and

$$(3.3) \quad d_\beta(w_1)v - d_\beta(v)w_1 = e^\gamma[(d_\beta(w_1)r_\gamma(v) - d_\beta(v)r_1) + d_\beta(w_1)a_\gamma(v)\varrho_X + d_\beta(w_1)\xi].$$

Since $d_\beta(w_1) > 0$ and $a_\gamma(v) \neq 0$, we have

$$\begin{aligned} (3.4) \quad 0 &= d_\beta(w_1)a_\gamma(v)(d_\beta(v_1)\xi - d_\beta(v)\xi_1) - (d_\beta(v_1)a_\gamma(v) - d_\beta(v)a_\gamma(v_1))d_\beta(w_1)\xi \\ &= d_\beta(w_1)d_\beta(v)(a_\gamma(v_1)\xi - a_\gamma(v)\xi_1). \end{aligned}$$

Hence the claim holds. \square

Corollary 3.1.2. *For the subobject F_1 of F in Lemma 3.1.1, we have*

$$(3.5) \quad r_\gamma(F_1)d_\gamma(F) - r_\gamma(F)d_\gamma(F_1) = \left(\text{rk } F_1 - \text{rk } F \frac{-r_1 a_\gamma(F_1)}{-r_1 a_\gamma(F)} \right) d_\gamma(F).$$

Proof. By Lemma 3.1.1,

$$v(F_1) = \text{rk } F_1 e^\gamma + a_\gamma(F_1) \left(\varrho_X + \frac{1}{a_\gamma(v)} (d_\gamma(v)H + D_\gamma(v) + (d_\gamma(v)H + D_\gamma(v), \gamma)\varrho_X) \right).$$

\square

3.2. Gieseker semi-stability and semi-stable objects in a chamber. From now on, we assume that (β, ω) belongs to exactly one wall W_{w_1} for v . Then there are two chambers \mathcal{C}_\pm which are adjacent to W_{w_1} in a neighborhood of (β, ω) :

$$\begin{aligned} (3.6) \quad \mathcal{C}_+ &:= \{(\beta', \omega') | \phi_{(\beta', \omega')}(v) - \phi_{(\beta', \omega')}(w_1) > 0\}, \\ \mathcal{C}_- &:= \{(\beta', \omega') | \phi_{(\beta', \omega')}(v) - \phi_{(\beta', \omega')}(w_1) < 0\}. \end{aligned}$$

We shall study the Bridgeland semi-stability for \mathcal{C}_\pm . This subsection is devoted to the study for \mathcal{C}_+ . The case \mathcal{C}_- is treated in the next subsection.

Lemma 3.2.1. *For a γ' -twisted stable sheaf E of $v(E) = \Phi(v)$ with respect to \widehat{L} , we set $F := \widehat{\Phi}(E)$. Assume that $d_\gamma(F) \neq 0$. We take ω_+ such that $(\beta, \omega_+) \in \mathcal{C}_+$, that is, $\phi_{(\beta, \omega_+)}(w_1) < \phi_{(\beta, \omega_+)}(F)$. Then F is a $\sigma_{(\beta, \omega_+)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$.*

Proof. We set $\phi_+ := \phi_{(\beta, \omega_+)}$ and $\phi := \phi_{(\beta, \omega)}$. Let F_1 be a subobject of F with $\phi(F_1) = \phi(F)$. Then F_1 and $F_2 := F/F_1$ are $\sigma_{(\beta, \omega)}$ -semi-stable objects of $\mathfrak{A}_{(\beta, \omega)}$ with the phase $\phi(F)$. Assume that F_1 is a $\sigma_{(\beta, \omega_+)}$ -semi-stable object with $\phi_+(F_1) > \phi_+(F)$. Since $\phi_+(F_1) > \phi_+(w_1)$, $\text{Hom}(F_1, \mathbf{E}_{|X \times \{x_1\}}) = 0$ for all $x_1 \in X_1$. Then Theorem 2.2.1 implies that $E_i := \Phi(F_i)$ ($i = 1, 2$) are μ -semi-stable torsion free sheaves on X_1 fitting in an exact sequence

$$(3.7) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0.$$

Since $\phi_+(F) > \phi_+(w_1)$ and $\phi(F) = \phi(w_1)$, $\Sigma_{(\beta, \omega_+)}(w_1, F) > 0 = \Sigma_{(\beta, \omega)}(w_1, F)$ implies that

$$(3.8) \quad \begin{aligned} -r_1 d_\gamma(F)(\omega_+^2) &= (r_\beta(F) d_\beta(w_1) - r_\beta(w_1) d_\beta(F))(\omega_+^2) \\ &< 2(a_\beta(F) d_\beta(w_1) - a_\beta(w_1) d_\beta(F)) \\ &= (r_\beta(F) d_\beta(w_1) - r_\beta(w_1) d_\beta(F))(\omega^2) = -r_1 d_\gamma(F)(\omega^2). \end{aligned}$$

Now we divide the argument into two cases.

(i) Assume that $r_1 d_\gamma(F) > 0$. We have $(\omega_+^2) > (\omega^2)$. Then $\phi_+(F_1) > \phi_+(F)$ implies that

$$(3.9) \quad \begin{aligned} &(r_\beta(F_1) d_\beta(F) - r_\beta(F) d_\beta(F_1))(\omega_+^2) \\ &< 2(a_\beta(F_1) d_\beta(F) - a_\beta(F) d_\beta(F_1)). \end{aligned}$$

Hence

$$r_\gamma(F_1) d_\gamma(F) - r_\gamma(F) d_\gamma(F_1) = r_\beta(F_1) d_\beta(F) - r_\beta(F) d_\beta(F_1) < 0.$$

Since $-r_1 \chi(E_1(-\gamma')) = \text{rk } F_1$ and $-r_1 \chi(E(-\gamma')) = \text{rk } F$, we get

$$(3.10) \quad \begin{aligned} 0 &> r_\gamma(F_1) d_\gamma(F) - r_\gamma(F) d_\gamma(F_1) = \left(\text{rk } F_1 - \text{rk } F \frac{-r_1 a_\gamma(F_1)}{-r_1 a_\gamma(F)} \right) d_\gamma(F) \\ &= -r_1 \left(\chi(E_1(-\gamma')) - \chi(E(-\gamma')) \frac{\text{rk } E_1}{\text{rk } E} \right) d_\gamma(F) \end{aligned}$$

by Corollary 3.1.2. Hence

$$\frac{\chi(E_1(-\gamma'))}{\text{rk } E_1} > \frac{\chi(E(-\gamma'))}{\text{rk } E},$$

which is a contradiction.

(ii) Assume that $r_1 d_\gamma(F) < 0$. Then

$$(3.11) \quad \begin{aligned} 0 &< r_\gamma(F_1) d_\gamma(F) - r_\gamma(F) d_\gamma(F_1) = \left(\text{rk } F_1 - \text{rk } F \frac{-r_1 a_\gamma(F_1)}{-r_1 a_\gamma(F)} \right) d_\gamma(F) \\ &= -r_1 \left(\chi(E_1(-\gamma)) - \chi(E(-\gamma')) \frac{\text{rk } E_1}{\text{rk } E} \right) d_\gamma(F). \end{aligned}$$

Hence

$$\frac{\chi(E_1(-\gamma'))}{\text{rk } E_1} > \frac{\chi(E(-\gamma'))}{\text{rk } E},$$

which is a contradiction.

Therefore $\phi_+(F_1) \leq \phi_+(F)$. Thus F is $\sigma_{(\beta, \omega_+)}$ -semi-stable. \square

In order to treat the case where $d_\gamma(v) = 0$, we need to choose $(\beta', \omega) \in \mathcal{C}_+$. We set $\gamma - \beta' := \lambda H + \nu'$, $(\nu, H) = 0$. We need the following claim.

Lemma 3.2.2. *If $d_\gamma(F) = d_\gamma(F_1) = 0$, then*

$$(3.12) \quad \begin{aligned} &a_{\beta'}(F) d_{\beta'}(F_1) - a_{\beta'}(F_1) d_{\beta'}(F) \\ &= \lambda ((a_\gamma(F) + (D_\gamma(F), \nu')) \text{rk } F_1 - (a_\gamma(F_1) + (D_\gamma(F_1), \nu')) \text{rk } F). \end{aligned}$$

Proof. It is a consequence of Lemma 1.3.1. \square

We take β' such that $(D_\gamma(F), \nu') > (D_\gamma(F), \nu)$. Since $d_\gamma(F) = d_\gamma(w_1) = 0$ and $\phi_{(\beta, \omega)}(F) = \phi_{(\beta, \omega)}(w_1)$, we have $\phi_{(\beta', \omega)}(F) > \phi_{(\beta', \omega)}(w_1)$.

Lemma 3.2.3. *For a γ' -twisted stable sheaf E of $v(E) = \Phi(v)$ with respect to \widehat{L} , we set $F := \widehat{\Phi}(E)$. Assume that $d_\gamma(F) = 0$ and take β' such that $(\beta', \omega) \in \mathcal{C}_+$, that is, $\phi_{(\beta', \omega)}(F) > \phi_{(\beta', \omega)}(w_1)$. Then F is a $\sigma_{(\beta', \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta', \omega)}$.*

Proof. Since $\phi_{(\beta', \omega)}(F) > \phi_{(\beta', \omega)}(w_1)$,

$$(3.13) \quad 0 = (r_{\beta'}(F)d_{\beta'}(w_1) - r_{\beta'}(w_1)d_{\beta'}(F)) \frac{(\omega^2)}{2} < r_1 \lambda(a_\gamma(F) + (D_\gamma(F), \nu')).$$

Let F_1 be a $\sigma_{(\beta', \omega)}$ -semi-stable subobject of F with $\phi_{(\beta, \omega)}(F_1) = \phi_{(\beta, \omega)}(F)$. Assume that $\phi_{(\beta', \omega)}(F_1) > \phi_{(\beta', \omega)}(F)$. Then we have the same exact sequence in (3.7). By Lemma 3.2.2,

$$(3.14) \quad \lambda((a_\gamma(F) + (D_\gamma(F), \nu')) \operatorname{rk} F_1 - (a_\gamma(F_1) + (D_\gamma(F_1), \nu')) \operatorname{rk} F) < 0.$$

By our choice of (β, ω) , Lemma 3.1.1 implies that $D_\gamma(F_1) = a_\gamma(F_1)D_\gamma(F)/a_\gamma(F)$. Hence

$$\lambda a_\gamma(F_1) (a_\gamma(F) + (D_\gamma(F), \nu')) \left(\frac{\operatorname{rk} F_1}{a_\gamma(F_1)} - \frac{\operatorname{rk} F}{a_\gamma(F)} \right) < 0.$$

Since $r_1 a_\gamma(F_1) < 0$ and $r_1 \lambda > 0$, we have

$$\frac{r_1^2 \chi(E_1(-\gamma'))}{\operatorname{rk} E_1} = \frac{\operatorname{rk} F_1}{a_\gamma(F_1)} > \frac{\operatorname{rk} F}{a_\gamma(F)} = \frac{r_1^2 \chi(E(-\gamma'))}{\operatorname{rk} E},$$

which is a contradiction. Therefore F is $\sigma_{(\beta', \omega)}$ -semi-stable. \square

The converse relation also holds:

Lemma 3.2.4. *We take $(\beta', \omega') \in \mathcal{C}_+$. Let F be a $\sigma_{(\beta', \omega')}$ -semi-stable object with $v(F) = v$. Then $\Phi(F)$ is a γ' -twisted semi-stable sheaf on X_1 .*

Proof. As in Lemma 3.2.1 and Lemma 3.2.3, we may assume that

$$(3.15) \quad (\beta', \omega') = \begin{cases} (\beta, \omega_+), & d_\gamma(v) \neq 0, \\ (\beta', \omega), & d_\gamma(v) = 0. \end{cases}$$

By Theorem 2.2.1, E is a μ -semi-stable sheaf on X_1 . If E is not γ' -twisted semi-stable, then there is an exact sequence

$$(3.16) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

with

$$(3.17) \quad \frac{(c_1(E_1), \widehat{L})}{\operatorname{rk} E_1} = \frac{(c_1(E), \widehat{L})}{\operatorname{rk} E}.$$

Applying Theorem 2.2.1 to E_1 and E_2 , we have an exact sequence

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

where $F_i := \widehat{\Phi}(E_i)$ are $\sigma_{(\beta, \omega)}$ -semi-stable objects of $\mathfrak{A}_{(\beta, \omega)}$ with $\phi(F_i) = \phi(w_1)$. Since $\phi_{(\beta', \omega')}(F) > \phi_{(\beta', \omega')}(w_1)$, in the same way as in the proof of Lemma 3.2.1 and Lemma 3.2.3, we see that

$$\frac{\chi(E_2(-\gamma'))}{\operatorname{rk} E_2} \geq \frac{\chi(E(-\gamma'))}{\operatorname{rk} E}.$$

Therefore E is γ' -twisted semi-stable. \square

3.3. Gieseker semi-stability and semi-stable objects in \mathcal{C}_- . Let us study the relation with $(-\gamma')$ -twisted semi-stability.

Lemma 3.3.1. *For a $(-\gamma')$ -twisted semi-stable sheaf E of $v(E) = \Phi(v)^\vee$ with respect to \widehat{L} , we set $F := \widehat{\Phi}(E^\vee)$.*

- (1) *Assume that $d_\gamma(F) \neq 0$. We take ω_- such that $(\beta, \omega_-) \in \mathcal{C}_-$, that is, $\phi_{(\beta, \omega_-)}(w_1) > \phi_{(\beta, \omega_-)}(F)$. Then F is a $\sigma_{(\beta, \omega_-)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega_-)}$.*
- (2) *Assume that $d_\gamma(F) = 0$. We take β' such that $(\beta', \omega) \in \mathcal{C}_-$, that is, $\phi_{(\beta', \omega)}(w_1) > \phi_{(\beta', \omega)}(F)$. Then F is a $\sigma_{(\beta', \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta', \omega)}$.*

Proof. (1) We set $\phi_- := \phi_{(\beta, \omega_-)}$ and $\phi := \phi_{(\beta, \omega)}$. We note that E^\vee fits in an exact triangle

$$\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X) \rightarrow E^\vee \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(E, \mathcal{O}_X)[-1] \rightarrow \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)[1],$$

where $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is a μ -semi-stable torsion free sheaf with (2.9) and $\mathcal{E}xt_{\mathcal{O}_X}^1(E, \mathcal{O}_X)$ is a 0-dimensional sheaf. Applying Theorem 2.2.1, we get F is a $\sigma_{(\beta, \omega)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ with $\phi(F) = \phi(w_1)$.

Assume that there is an exact sequence

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

such that F_2 is a semi-stable object with respect to (β, ω_-) such that $\phi_-(F_1) > \phi_-(F) > \phi_-(F_2)$ and $\phi(F_1) = \phi(F_2) = \phi(w_1)$. Then $\operatorname{Hom}(F_2, \mathbf{E}_{|X \times \{x_1\}}[k]) = 0$ for $k \geq 2$. Since $\operatorname{Hom}(F_2, \mathbf{E}_{|X \times \{x_1\}}[k]) = 0$ for

$k < 0$ and $\text{Hom}(F_2, \mathbf{E}_{|X \times \{x_1\}}) = 0$ except finitely many point $x_1 \in X_1$, we find that $E_1 := \Phi(F_1)^\vee$ and $E_2 := \Phi(F_2)^\vee$ are torsion free sheaves and there exists an exact sequence

$$0 \rightarrow E_2 \rightarrow E \rightarrow E_1 \rightarrow 0.$$

Since $\phi_-(F_2) < \phi_-(F) < \phi_-(w_1)$, we have

$$\begin{aligned} -r_1 d_\gamma(F)(\omega_-^2) &= (r_\beta(F) d_\beta(w_1) - r_\beta(w_1) d_\beta(F))(\omega_-^2) \\ &> 2(a_\beta(F) d_\beta(w_1) - a_\beta(w_1) d_\beta(F)) \\ &= (r_\beta(F) d_\beta(w_1) - r_\beta(w_1) d_\beta(F))(\omega^2) = -r_1 d_\gamma(F)(\omega^2). \end{aligned} \quad (3.18)$$

We divide the rest argument into two cases.

(i) If $r_1 d_\gamma(F) > 0$, then we have $(\omega_-^2) < (\omega^2)$. Then

$$\begin{aligned} &(r_\beta(F_2) d_\beta(F) - r_\beta(F) d_\beta(F_2))(\omega_-^2) \\ &> 2(a_\beta(F_2) d_\beta(F) - a_\beta(F) d_\beta(F_2)) \end{aligned} \quad (3.19)$$

implies

$$r_\gamma(F_2) d_\gamma(F) - r_\gamma(F) d_\gamma(F_2) = r_\beta(F_2) d_\beta(F) - r_\beta(F) d_\beta(F_2) < 0.$$

By Lemma 1.4.2, $\chi(E^\vee(-\gamma')) = -r_\gamma(F)/r_1$ and $\chi(\Phi(F_2)(-\gamma')) = -r_\gamma(F_2)/r_1$. Hence

$$\begin{aligned} 0 &> r_\gamma(F_2) d_\gamma(F) - r_\gamma(F) d_\gamma(F_2) \\ &= \left(\text{rk } F_2 - \text{rk } F \frac{-r_1 a_\gamma(F_2)}{-r_1 a_\gamma(F)} \right) d_\gamma(F) \\ &= -r_1 d_\gamma(F) \left(\chi(E_2(\gamma')) - \chi(E(\gamma)) \frac{-r_1 a_\gamma(F_2)}{-r_1 a_\gamma(F)} \right). \end{aligned} \quad (3.20)$$

Hence

$$\frac{\chi(E_2(\gamma'))}{\text{rk } E_2} = \frac{\chi(\Phi(F_2)(-\gamma'))}{\text{rk } \Phi(F_2)} > \frac{\chi(\Phi(F)(-\gamma'))}{\text{rk } \Phi(F)} = \frac{\chi(E(\gamma'))}{\text{rk } E},$$

which is a contradiction.

(ii) If $r_1 d_\gamma(F) < 0$, then we have $(\omega_-^2) > (\omega^2)$. Then

$$\begin{aligned} &(r_\beta(F_2) d_\beta(F) - r_\beta(F) d_\beta(F_2))(\omega_-^2) \\ &> 2(a_\beta(F_2) d_\beta(F) - a_\beta(F) d_\beta(F_2)) \end{aligned} \quad (3.21)$$

implies

$$r_\gamma(F_2) d_\gamma(F) - r_\gamma(F) d_\gamma(F_2) = r_\beta(F_2) d_\beta(F) - r_\beta(F) d_\beta(F_2) > 0.$$

By Lemma 1.4.2, we have

$$\begin{aligned} 0 &< r_\gamma(F_2) d_\gamma(F) - r_\gamma(F) d_\gamma(F_2) \\ &= \left(\text{rk } F_2 - \text{rk } F \frac{-r_1 a_\gamma(F_2)}{-r_1 a_\gamma(F)} \right) d_\gamma(F) \\ &= -r_1 d_\gamma(F) \left(\chi(E_2(\gamma')) - \chi(E(\gamma)) \frac{-r_1 a_\gamma(F_2)}{-r_1 a_\gamma(F)} \right). \end{aligned} \quad (3.22)$$

Hence

$$\frac{\chi(E_2(\gamma'))}{\text{rk } E_2} = \frac{\chi(\Phi(F_2)(-\gamma'))}{\text{rk } \Phi(F_2)} > \frac{\chi(\Phi(F)(-\gamma'))}{\text{rk } \Phi(F)} = \frac{\chi(E(\gamma'))}{\text{rk } E},$$

which is a contradiction.

Therefore F is a $\sigma_{(\beta, \omega_-)}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega_-)}$.

(2) Assume that $d_\gamma(F) = 0$. For $(\beta', \omega) \in \mathcal{C}_-$, we have

$$0 = (r_{\beta'}(F) d_{\beta'}(w_1) - r_{\beta'}(w_1) d_{\beta'}(F)) \frac{(\omega^2)}{2} > r_1 \lambda(a_\gamma(F) + (D_\gamma(F), \nu')).$$

Since $\phi_{(\beta', \omega)}(F_2) < \phi_{(\beta', \omega)}(F)$, we have

$$\lambda((a_\gamma(F) + (D_\gamma(F), \nu')) \text{rk } F_2 - (a_\gamma(F_2) + (D_\gamma(F_2), \nu')) \text{rk } F) > 0.$$

By the choice of (β', ω) , $D_\gamma(F_2) = a_\gamma(F_2) D_\gamma(F)/a_\gamma(F)$. Hence

$$\lambda a_\gamma(F_2)(a_\gamma(F) + (D_\gamma(F), \nu')) \left(\frac{\text{rk } F_2}{a_\gamma(F_2)} - \frac{\text{rk } F}{a_\gamma(F)} \right) > 0.$$

Therefore

$$\frac{\text{rk } F_2}{a_\gamma(F_2)} > \frac{\text{rk } F}{a_\gamma(F)}.$$

□

The converse relation also holds:

Lemma 3.3.2. *We take $(\beta', \omega') \in \mathcal{C}_-$. Let F be a $\sigma_{(\beta', \omega')}$ -semi-stable object of $\mathfrak{A}_{(\beta, \omega)}$ with $v(F) = v$. Then $\Phi(F)^\vee$ is a $(-\gamma')$ -twisted semi-stable sheaf on X_1 .*

Proof. We may take the same (β', ω') in Lemma 3.3.1. By Theorem 2.2.1, $E := \Phi(F)^\vee$ is a μ -semi-stable torsion free sheaf on X_1 . Assume that E is not $(-\gamma')$ -twisted semi-stable. Then there is an exact sequence

$$(3.23) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that E_i , $i = 1, 2$ are torsion free sheaves with

$$(3.24) \quad \frac{(c_1(E_i), \widehat{L})}{\text{rk } E_i} = \frac{(c_1(E), \widehat{L})}{\text{rk } E}.$$

Applying Theorem 2.2.1 to E_i^\vee , we have an exact sequence

$$(3.25) \quad 0 \rightarrow F_2 \rightarrow F \rightarrow F_1 \rightarrow 0,$$

where $F_i := \widehat{\Phi}(E_i^\vee)$ are $\sigma_{(\beta, \omega)}$ -semi-stable objects of $\mathfrak{A}_{(\beta, \omega)}$ with $\phi_{(\beta, \omega)}(F_i) = \Phi_{(\beta, \omega)}(w_1)$. By using the computations in the proof of Lemma 3.3.1, it is easy to see that F is $\sigma_{(\beta', \omega')}$ -semi-stable if and only if E is $(-\gamma')$ -twisted semi-stable. \square

Summarizing our argument, we have

Theorem 3.3.3. *Let X be an abelian surface, or a K3 surface with $\text{NS}(X) = \mathbb{Z}H$. Assume that there is a smooth projective surface X_1 which is the moduli space $M_{(\beta, \omega)}(w_1)$. Let v be a Mukai vector with $d_\beta(v) > 0$. Assume that (β, ω) satisfies*

$$(3.26) \quad \langle d_\beta(v)w_1 - d_\beta(w_1)v, e^{\beta + \sqrt{-1}\omega} \rangle = 0.$$

If (β, ω) does not belong to any wall for v , or w_1 defines a wall W_{w_1} for v and (β, ω) belongs to exactly one wall W_{w_1} , then $\mathcal{M}_{(\beta', \omega')}(v) \cong \mathcal{M}_{\widehat{L}}(u)^{ss}$, where $u = \Phi(v)$ for $(\beta', \omega') \in \mathcal{C}_+$ and $u = \Phi(v)^\vee$ for $(\beta', \omega') \in \mathcal{C}_-$. In particular, there is a coarse moduli scheme $M_{(\beta', \omega')}(v)$ and $M_{(\beta', \omega')}(v) \cong \overline{M}_{\widehat{L}}(u)$.

Remark 3.3.4. Let E be an α -twisted sheaf on X_1 such that $v(E) = w$ is a primitive Mukai vector. Assume that \mathbf{F} is a family of $(-\alpha)$ -twisted sheaves consisting of $(-\gamma')$ -twisted stable sheaves of Mukai vector $r_1 e^{-\gamma'}$ with respect to \widehat{L} . Then for a sufficiently large n , E is a stable α -twisted sheaf on X_1 if and only if $\Phi_{X_1 \rightarrow X}^{\mathbf{F}}(E(n\widehat{L}))$ is a stable sheaf on X . In particular, $\mathcal{M}_{\widehat{L}}^{\gamma'}(w)^{ss}$ is isomorphic to $\mathcal{M}_{\widehat{L}}^\gamma(u)^{ss}$, where $u := v(\Phi_{X_1 \rightarrow X}^{\mathbf{E}}(E(n\widehat{L})))$. Moreover $\mathcal{M}_{\widehat{L}}^\gamma(u)^{ss}$ consists of μ -stable sheaves. Then the second cohomology group of $M_{(\beta', \omega')}(v)$ is described by v^\perp and the albanese map $M_{(\beta', \omega')}(v) \rightarrow X \times \widehat{X}$.

3.4. The projectivity of $M_{(\beta, \omega)}(w_1)$ for a K3 surface. Fix a primitive isotropic Mukai vector $w_1 := r_1 e^\gamma$ with $d_\beta(w_1) > 0$. In this subsection, let us study the assumption on $M_{(\beta, \omega)}(w_1)$ in Theorem 3.3.3.

Lemma 3.4.1. *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$.*

- (1) *There is at most one Mukai vector v such that $\langle v^2 \rangle = -2$ and $\phi_{(\beta, \omega)}(v) = \phi_{(\beta, \omega)}(w_1)$.*
- (2) *There is a unique primitive isotropic Mukai vector v such that $\phi_{(\beta, \omega)}(v) = \phi_{(\beta, \omega)}(w_1)$ and $\langle v, w_1 \rangle \neq 0$.*

Proof. We set $u_0 := w_1$. We take a primitive Mukai vector $u_1 \in H^*(X, \mathbb{Z})_{\text{alg}}$ such that

$$(3.27) \quad u_0^\perp \cap H^*(X, \mathbb{Z})_{\text{alg}} = \mathbb{Z}u_0 \oplus \mathbb{Z}u_1 \subset \mathbb{Q}e^\gamma \oplus \mathbb{Q}(H + (H, \gamma)\varrho_X).$$

We next take $u_2 \in H^*(X, \mathbb{Z})_{\text{alg}}$ such that

$$(3.28) \quad -\langle u_2, w_1 \rangle = \min\{-\langle u, w_1 \rangle > 0 \mid u \in H^*(X, \mathbb{Z})_{\text{alg}}\}.$$

Since $H^*(X, \mathbb{Z})_{\text{alg}}$ is generated by $u_0^\perp \cap H^*(X, \mathbb{Z})_{\text{alg}}$ and u_2 ,

$$H^*(X, \mathbb{Z})_{\text{alg}} = \mathbb{Z}u_0 \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2.$$

- (1) If $v = \sum_{i=0}^2 x_i u_i$, $x_i \in \mathbb{Z}$ is a (-2) -vector, then we get

$$-2 = \langle (x_1 u_1 + x_2 u_2)^2 \rangle + 2\langle x_1 u_1 + x_2 u_2, x_0 u_0 \rangle,$$

which implies that $\gcd(x_1, x_2) = 1$. On the other hand, if $\phi_{(\beta, \omega)}(v) = \phi_{(\beta, \omega)}(u_0)$, then Lemma 2.1.3 implies that $-r_1(c_1(v(-\gamma)), L) + \lambda\langle v, u_0 \rangle = 0$. Hence

$$\lambda x_2 \langle u_2, u_0 \rangle = r_1(x_1(c_1(u_1(-\gamma)), L) + x_2(c_1(u_2(-\gamma)), L)),$$

which implies that x_1/x_2 is determined. Thus v is unique up to sign. Since $d_\beta(v) > 0$, v is unique. \square

The proof of (2) is similar. \square

Proposition 3.4.2. *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. For $\gamma \in \mathbb{Q}H$, let $w_1 := r_1 e^\gamma$ be the primitive isotropic Mukai vector with $d_\beta(w_1) > 0$. Let v be the unique primitive isotropic Mukai vector such that $\phi_{(\beta,\omega)}(v) = \phi_{(\beta,\omega)}(w_1)$ and $\langle v, w_1 \rangle \neq 0$.*

- (1) *Assume that $\mathcal{M}_{(\beta,\omega)}(w_1)$ contains a properly $\sigma_{(\beta,\omega)}$ -semi-stable object.*
 - (i) *There is a $\sigma_{(\beta,\omega)}$ -stable object E_1 such that $\langle v(E_1)^2 \rangle = -2$ and $\phi_{(\beta,\omega)}(v(E_1)) = \phi_{(\beta,\omega)}(w_1)$.*
 - (ii) *$\langle w_1, v(E_1) \rangle < 0$ and $v = w_1 + \langle w_1, v(E_1) \rangle v(E_1)$.*
 - (iii) *$\mathcal{M}_{(\beta,\omega)}(v)$ consists of stable objects.*
- (2) *If there is no (-2) -vector with the phase $\phi_{(\beta,\omega)}(w_1)$, then $\mathcal{M}_{(\beta,\omega)}(w_1)$ and $\mathcal{M}_{(\beta,\omega)}(v)$ consist of stable objects.*

Proof. (1) If (β, ω) belongs to a wall, then we take an object E of $\mathcal{M}_{(\beta,\omega)}(w_1)$ such that E is not $\sigma_{(\beta,\omega')}$ -semi-stable, where ω' is sufficiently close to ω . We take the Harder-Narasimhan filtration of E with respect to (β, ω') :

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E.$$

Then $\phi_{(\beta,\omega)}(F_i/F_{i-1}) = \phi_{(\beta,\omega)}(E)$ and $\langle v(F_i/F_{i-1})^2 \rangle \leq 0$. By Lemma 3.4.1, there are primitive Mukai vectors $v_1, v_2 \in H^*(X, \mathbb{Z})_{\text{alg}}$ such that

$$\phi_{(\beta,\omega)}(v_i) = \phi_{(\beta,\omega)}(w_1), \quad \langle v_1^2 \rangle = 0, \quad \langle v_2^2 \rangle = -2.$$

Since $v(F_i/F_{i-1}) \in \mathbb{Z}v_1$ or $v(F_i/F_{i-1}) \in \mathbb{Z}v_2$, we have

$$s = 2, \quad \{v(F_1), v(F_2/F_1)\} = \{n_1 v_1, n_2 v_2\}, \quad n_1, n_2 > 0.$$

Then $0 = \langle (n_1 v_1 + n_2 v_2)^2 \rangle = 2n_2(n_1 \langle v_1, v_2 \rangle - n_2)$. Hence $n_2 = n_1 \langle v_1, v_2 \rangle$. Since $w_1 = v(E) = n_1 v_1 + n_2 v_2$ is primitive, $n_1 = 1$ and $w_1 = v_1 + \langle v_1, v_2 \rangle v_2$. Since $n_2 = n_1 \langle v_1, v_2 \rangle$, $\langle w_1, v_1 \rangle = -\langle v_2, v_1 \rangle < 0$. Let E_1 be the $\sigma_{(\beta,\omega)}$ -stable object with $v(E_1) = v_2$. Then (i) holds. Since $v = v_1$, (ii) holds. Since $\langle v, v(E_1) \rangle > 0$, the assumption of (1) does not hold for $\mathcal{M}_{(\beta,\omega)}(v)$. Thus (iii) holds.

By the proof of (1), (2) also follows. \square

The following is a special case of [5, Thm. 6.12], if the moduli space is fine.

Proposition 3.4.3. *Let X be a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. Let v be a primitive isotropic Mukai vector with $d_\beta(v) > 0$. If (β, ω) does not belong to any wall for v , then there is a coarse moduli space $M_{(\beta,\omega)}(v)$ which is a K3 surface. Moreover there is a Fourier-Mukai transform $\Psi : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ inducing an isomorphism $\mathcal{M}_{(\beta,\omega)}(v) \rightarrow \mathcal{M}_H(\Psi(v))^{ss}$.*

Proof. By [9, Prop. 1.6.10], we may assume that $\mathfrak{A}_{(\beta,\omega)} = \mathfrak{A}_{(\beta,tH)}$, $t > 1$. If $(\omega^2) \gg 0$, then the claim is obvious by [9, Cor. 2.2.9]. Hence it is sufficient to show that the claims are preserved under the wall-crossing. Assume that (β, ω) belongs to a wall W and the claim holds for (β, ω_+) with $(\omega_+^2) > (\omega^2)$. In a neighborhood of ω , we take ω_- with $(\omega_-^2) < (\omega^2)$. By Proposition 3.4.2 (1) (i), there is a $\sigma_{(\beta,\omega)}$ -stable object E_1 with $\phi_{(\beta,\omega)}(E_1) = \phi_{(\beta,\omega)}(v)$ and $\langle v(E_1)^2 \rangle = -2$. We note that E_1 satisfies $\text{Hom}(E_1, E_1) = \mathbb{k}$ and $\text{Hom}(E_1, E_1[p]) = 0$ ($p \neq 0, 2$). Then we have an autoequivalence $\Phi_{E_1} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ ([9, sect. 1]). For $E \in \mathcal{M}_{(\beta,\omega_+)}(v)$, by Proposition 3.4.2 (1) (ii), we have $\langle v, v(E_1) \rangle < 0$.

We first assume that $\phi_{(\beta,\omega_+)}(E_1) < \phi_{(\beta,\omega_+)}(v)$. Then $\text{Hom}(E, E_1) = 0$. By the proof of Proposition 3.4.2, we have an exact sequence

$$0 \rightarrow E_1^{\oplus n} \rightarrow E \rightarrow F \rightarrow 0$$

where $n = -\langle v, v(E_1) \rangle$ and F is a $\sigma_{(\beta,\omega)}$ -stable object with isotropic Mukai vector. Then we see that $\text{Ext}^1(E_1, E) = 0$ and $\Phi_{E_1}(E) = F$. Applying Φ_{E_1} again, we have an exact sequence

$$0 \rightarrow F \rightarrow \Phi_{E_1}(F) \rightarrow \text{Ext}^1(E_1, F) \otimes E_1 \rightarrow 0.$$

Then we see that $E' := \Phi_{E_1}(F)$ is a $\sigma_{(\beta,\omega_-)}$ -semi-stable object. Conversely for a $\sigma_{(\beta,\omega_-)}$ -semi-stable object E' , we have an exact sequence

$$0 \rightarrow F \rightarrow E' \rightarrow E_1^{\oplus n} \rightarrow 0$$

where $n = -\langle v, v(E_1) \rangle$ and F is a $\sigma_{(\beta,\omega)}$ -stable object. Then we see that $n = -\langle v, v(E_1) \rangle$ and $\Phi_{E_1}^{-1}(E') = F$. Moreover $\Phi_{E_1}^{-1}(F) \in \mathcal{M}_{(\beta,\omega_+)}(v)$. Therefore we have a sequence of isomorphisms

$$\mathcal{M}_{(\beta,\omega_+)}(v) \xrightarrow{\Phi_{E_1}} \mathcal{M}_{(\beta,\omega)}(u) \xrightarrow{\Phi_{E_1}} \mathcal{M}_{(\beta,\omega_-)}(v),$$

where $u = v + \langle v, v(E_1) \rangle v(E_1)$. If $\phi_{(\beta,\omega_+)}(E_1) > \phi_{(\beta,\omega_+)}(v)$, then by a similar argument, we have a sequence of isomorphisms

$$\mathcal{M}_{(\beta,\omega_+)}(v) \xrightarrow{\Phi_{E_1}^{-1}} \mathcal{M}_{(\beta,\omega)}(u) \xrightarrow{\Phi_{E_1}^{-1}} \mathcal{M}_{(\beta,\omega_-)}(v).$$

Therefore the claims hold for (β, ω_-) . \square

4. APPLICATIONS

4.1. The projectivity of some moduli spaces. In this subsection, we assume that X is an abelian surface or a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. Let

$$v := re^\beta + a_\beta \varrho_X + (d_\beta H + D) + (d_\beta H + D, \beta) \varrho_X, \quad D \in \text{NS}(X)_\mathbb{Q} \cap H^\perp$$

be a Mukai vector with $d_\beta(v) > 0$ and $\langle v^2 \rangle > 0$. Then $d_\beta > \frac{2a_\beta}{(H^2)d_\beta}r$. Assume that

$$w_1 := r_1 e^{\beta + \frac{d_1}{r_1}H} = r_1 e^\beta + d_1(H + (H, \beta) \varrho_X) + a_1 \varrho_X.$$

Then $a_1 = \frac{d_1^2(H^2)}{2r_1}$. Hence

$$(4.1) \quad \frac{d_1 a_\beta - d_\beta a_1}{d_1 r - d_\beta r_1} = \frac{\frac{d_1}{r_1} \left(a_\beta - d_\beta \frac{(H^2)d_1}{2r_1} \right)}{\frac{d_1}{r_1} r - d_\beta}.$$

We set $f(x) := \frac{x \left(a_\beta - d_\beta \frac{(H^2)}{2} x \right)}{xr - d_\beta}$, $x \in \mathbb{R}$. Then $f(x)$ defines a bijection $f : D \rightarrow \mathbb{R}_{>0}$, where

$$(4.2) \quad D := \begin{cases} (x_0, \frac{d_\beta}{r}), & r > 0 \\ (x_0, \infty), & r \leq 0, \end{cases}$$

$x_0 := \max\{\frac{2a_\beta}{(H^2)d_\beta}, 0\}$. For $x \in D$, we take a unique element $\omega_x \in \mathbb{R}_{>0}H$ such that $\frac{(\omega_x^2)}{2} = f(x)$. We define an injective map

$$(4.3) \quad \begin{array}{ccc} \iota_\beta : \mathbb{R}_{>0}H & \rightarrow & \mathfrak{H}_\mathbb{R} \\ \omega & \mapsto & (\eta, \omega), \end{array}$$

where $\beta = bH + \eta$, $\eta \in H^\perp \cap \text{NS}(X)_\mathbb{Q}$. Let $I \subset \mathbb{R}_{>0}H$ be the pull-back of a chamber in $\mathfrak{H}_\mathbb{R}$ by ι_β . We set

$$J := \{x \in \mathbb{R} \mid \omega_x \in I\}.$$

We take a rational number $\lambda \in J$. Then $\phi_{(\beta, \omega_\lambda)}(w_1) = \phi_{(\beta, \omega_\lambda)}(v)$ and ω_λ belongs to the same chamber as that of ω , where $w_1 = r_1 e^{\beta + \lambda H}$ is a primitive Mukai vector with $r_1 > 0$. Hence $\mathcal{M}_{(\beta, \omega)}(v)^{ss} = \mathcal{M}_{(\beta, \omega_\lambda)}(v)^{ss}$. Replacing w_1 if necessary, we have a primitive isotropic Mukai vector w_1 such that $\phi_{(\beta, \omega_\lambda)}(w_1) = \phi_{(\beta, \omega_\lambda)}(v)$ and $X_1 := M_{(\beta, \omega_\lambda)}(w_1)$ is a smooth projective surface (Proposition 3.4.2). Let Φ be the Fourier-Mukai transform in §2. Applying Corollary 2.2.3, we have an isomorphism $\mathcal{M}_{(\beta, \omega_\lambda)}(v) \rightarrow \mathcal{M}_{\widehat{H}}(u)^{ss}$, where $u = \Phi(v)$.

Theorem 4.1.1. *Let X be an abelian surface or a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. Assume that (β, ω) is general. There is a coarse moduli scheme $M_{(\beta, \omega)}(v)$ which is isomorphic to the projective scheme $\overline{M}_{\widehat{H}}(u)$, where $u = \Phi(v)$.*

In particular, the moduli spaces in [1] are projective, if ω is general.

Remark 4.1.2. Maciocia and Meachan showed the claim for $v = 1 + 2H + n\varrho_X$, where X is an abelian surface with $\text{NS}(X) = \mathbb{Z}H$ in [8, Thm. 3.1]. It is easy to see that their proof also works for any v and get the same result for abelian surfaces.

4.2. The dependence of walls on β . We shall study the structure of walls for stabilities under the deformation of β . In this subsection, we assume that X is an abelian surface. Let us start with the following lemma.

Lemma 4.2.1. *Assume that non-zero vectors*

$$v_i := r_i e^\beta + a_i \varrho_X + d_i H + D_i + (d_i H + D_i, \beta) \varrho_X, \quad D_i \in H^\perp \cap \text{NS}(X)_\mathbb{Q} \quad (i = 1, 2)$$

satisfy (1) $\langle v_i^2 \rangle \geq 0$ and (2) $Z_{(\beta, \omega)}(v_1)$ and $Z_{(\beta, \omega)}(v_2)$ are linearly dependent over \mathbb{R} . Then $d_1 d_2 \langle v_1, v_2 \rangle > 0$ or $d_1 = d_2 = 0$.

Proof. Since $Z_{(\beta, \omega)}(v_1)$ and $Z_{(\beta, \omega)}(v_2)$ are linearly dependent, we have

$$(d_1 r_2 - d_2 r_1) \frac{(\omega^2)}{2} = (d_1 a_2 - d_2 a_1).$$

By [9, Lem. 3.1], we have

$$(4.4) \quad \begin{aligned} \langle v_1, v_2 \rangle (d_1 d_2) &= -\frac{1}{2}((d_1 D_2 - d_2 D_1)^2) + \frac{d_2^2 \langle v_1^2 \rangle}{2} + \frac{d_1^2 \langle v_2^2 \rangle}{2} + (d_1 r_2 - d_2 r_1)(d_1 a_2 - d_2 a_1) \\ &= -\frac{1}{2}((d_1 D_2 - d_2 D_1)^2) + \frac{d_2^2 \langle v_1^2 \rangle}{2} + \frac{d_1^2 \langle v_2^2 \rangle}{2} + (d_1 r_2 - d_2 r_1)^2 \frac{(\omega^2)}{2} \geq 0. \end{aligned}$$

If the equality holds, then $d_1 r_2 - d_2 r_1 = d_1 a_2 - d_2 a_1 = d_1 D_2 - d_2 D_1 = 0$. Thus $d_1 v_2 = d_2 v_1$. If $d_1 \neq 0$ or $d_2 \neq 0$, then $v_1 = 0$ or $v_2 = 0$, which is a contradiction. Therefore the claim holds. \square

The following characterization of the walls for stabilities is a consequence of the Bogomolov inequality.

Proposition 4.2.2. *Assume that $\langle v^2 \rangle > 0$. For a Mukai vector v_1 , we set $v_2 := v - v_1$. Then $v_1 \notin \mathbb{Q}v$ defines a wall in $\mathfrak{H}_{\mathbb{R}}$, if and only if (1) $\langle v_1^2 \rangle, \langle v_2^2 \rangle \geq 0$ and (2) $\langle v_1, v_2 \rangle > 0$.*

Proof. We write

$$\begin{aligned} v &= re^\beta + a\varrho_X + dH + D + (dH + D, \beta)\varrho_X, \quad D \in H^\perp \cap \text{NS}(X)_{\mathbb{Q}}, \\ v_i &= r_i e^\beta + a_i \varrho_X + d_i H + D_i + (d_i H + D_i, \beta)\varrho_X, \quad D_i \in H^\perp \cap \text{NS}(X)_{\mathbb{Q}} \quad (i = 1, 2). \end{aligned}$$

(I) Assume that v_1 defines a wall, that is, there are $\sigma_{(\beta, \omega)}$ -semi-stable objects E_i ($i = 1, 2$) of $\mathfrak{A}_{(\beta, \omega)}$ with $v(E_i) = v_i$, and $Z_{(\beta, \omega)}(E_i)$ ($i = 1, 2$) are linearly dependent over \mathbb{R} . Then $d_1, d_2 \geq 0$. By Lemma 4.2.1, we have (i) $d_1 d_2 \langle v_1, v_2 \rangle > 0$ or (ii) $d_1 = d_2 = 0$. In the first case, $d \geq 0$ implies that $d_1, d_2 > 0$. Hence $\langle v_1, v_2 \rangle > 0$. So it is enough to consider the second case. In this case, $d_1 = d_2 = 0$ implies that

$$(4.5) \quad r_1 r_2 \langle v_1, v_2 \rangle = \frac{r_2^2 \langle v_1^2 \rangle}{2} + \frac{r_1^2 \langle v_2^2 \rangle}{2} - \frac{1}{2}((r_2 D_1 - r_1 D_2)^2) \geq 0.$$

If the equality holds, then $r_2 D_1 - r_1 D_2 = 0$. Since $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{R}_{<0}$ by (1.2), the condition $d_1 = d_2 = 0$ and the definition of semi-stable object, we have $a_i - r_i \frac{(\omega^2)}{2} > 0$ ($i = 1, 2$). By $0 \leq \langle v_i^2 \rangle = -2r_i a_i + (D_i^2) \leq -2r_i a_i$, $-r_i \geq 0$ and $a_i \geq 0$ ($i = 1, 2$). If $r_1 = r_2 = 0$, then $r = 0$ and $\langle v^2 \rangle \leq 0$. Therefore $r_1 \neq 0$ or $r_2 \neq 0$. If $r_1 \neq 0$ and $r_2 = 0$, then $v_2 = a_2 \varrho_X \neq 0$ and $\langle v_1, v_2 \rangle = -r_1 a_2 > 0$. If $r_1, r_2 \neq 0$ and $r_1 r_2 \langle v_1, v_2 \rangle = 0$, then $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 0$ and $D_1/r_1 = D_2/r_2$. Hence $v_i = r_i e^{D_i/r_i}$ ($i = 1, 2$), which implies v is isotropic. Therefore if $r_1, r_2 \neq 0$, then $r_1 r_2 \langle v_1, v_2 \rangle > 0$. Hence $r_1, r_2 < 0$ and $\langle v_1, v_2 \rangle > 0$.

(II) Conversely assume that $\langle v_1^2 \rangle, \langle v_2^2 \rangle \geq 0$ and $\langle v_1, v_2 \rangle > 0$. For (β, ω) with $Z_{(\beta, \omega)}(v) \in \mathbb{H} \cup \mathbb{R}_{<0}$, assume that $Z_{(\beta, \omega)}(v_i)$ ($i = 1, 2$) are linearly dependent over \mathbb{R} . We shall show that $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Then there are $\sigma_{(\beta, \omega)}$ -semi-stable objects E_i ($i = 1, 2$) of $\mathfrak{A}_{(\beta, \omega)}$ with $v(E_i) = v_i$. This means that v_1 defines a wall for v .

We first assume that $d > 0$. In this case, Lemma 4.2.1 implies that $d_1, d_2 > 0$. Hence we get $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

If $d = 0$, then we have $0 < \langle v^2 \rangle = -2ra + (D^2) \leq -2ra$ and $-Z_{(\beta, \omega)}(v) = -r \frac{(\omega^2)}{2} + a > 0$. Hence $-r, a > 0$. By Lemma 4.2.1, $d = d_1 + d_2$ and $\langle v_1, v_2 \rangle > 0$, we have $d_1 = d_2 = 0$. Then by (4.5), we have $r_1 r_2 \langle v_1, v_2 \rangle > 0$ or $r_1 r_2 = 0$. For the first case $r_1 r_2 \langle v_1, v_2 \rangle > 0$, our assumption implies that $r_1 r_2 > 0$. Since $-r_1 a_1 \geq 0$ and $-r_2 a_2 \geq 0$, $-a_1/r_1, -a_2/r_2 \geq 0$. Since

$$(4.6) \quad \begin{aligned} 0 &< -r \frac{(\omega^2)}{2} + a \\ &= -r_1 \left(\frac{(\omega^2)}{2} - \frac{a_1}{r_1} \right) - r_2 \left(\frac{(\omega^2)}{2} - \frac{a_2}{r_2} \right), \end{aligned}$$

we have $-r_1, -r_2 > 0$. Therefore $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{R}_{<0}$. For the second case $r_1 r_2 = 0$, we may assume that $r_1 = r < 0$ and $r_2 = 0$. Then we see that $v_2 = a_2 \varrho_X$. Since $\langle v_1, v_2 \rangle = -r_1 a_2 > 0$ and $r_1 < 0, a_2 > 0$. Since $0 \leq \langle v_1^2 \rangle \leq -2r_1 a_1, a_1 \geq 0$. Therefore $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{R}_{<0}$. \square

We set $\beta := \beta_0 + sH$, $s \leq d_{\beta_0}/r$. Then $d(s) := d_\beta(v)$ and $d_i(s) := d_\beta(v_i)$ ($i = 1, 2$) are function of s . We note that the conditions in Lemma 4.2.1 are independent of s . By Lemma 4.2.1, we have the following.

Lemma 4.2.3. *Assume that $\langle v_1^2 \rangle, \langle v_2^2 \rangle \geq 0$. We take (s, t) in*

$$C := \{(s, t) \mid \Sigma_{(\beta, tH)}(v, v_1) = 0, t > 0, d(s) \geq 0\}.$$

Then the following conditions are equivalent:

- (1) $0 < d_1(s) < d(s)$ at a point $(s, t) \in C$.
- (2) $0 < d_1(s) < d(s)$ for all $(s, t) \in C$ with $d(s) > 0$.
- (3) $\langle v_1, v_2 \rangle > 0$.

Remark 4.2.4. Keep the notation as above. We note that $r_1 d_2(s) - r_2 d_1(s)$ does not depend on the choice of s . Assume that the constant $r_1 d_2(s) - r_2 d_1(s) \neq 0$ and $\langle v_1, v_2 \rangle > 0$. Then $d_1(s) d_2(s) \langle v_1, v_2 \rangle > 0$ and $d(s) \neq 0$ for all $(s, t) \in C$.

Indeed if $d_1(s) = d_2(s) = 0$, then $r_1 d_2(s) - r_2 d_1(s) = 0$. By Lemma 4.2.1, we have $d_1(s) d_2(s) \langle v_1, v_2 \rangle > 0$. Since $\langle v_1, v_2 \rangle > 0, d_1(s) d_2(s) > 0$, which implies that $d(s) \neq 0$.

For an isotropic Mukai vector

$$w_1 = r_1 e^{\beta_0 + xH} = r_1 e^{\beta + (x-s)H},$$

we see that

$$\begin{aligned}
(4.7) \quad & \frac{(x-s) \left(a_{\beta_0+sH} - d_{\beta_0+sH} (x-s) \frac{(H^2)}{2} \right)}{(x-s)r - (d_{\beta_0} - rs)} \\
&= \frac{(x-s) \left(a_{\beta_0} - d_{\beta_0}s(H^2) + rs^2 \frac{(H^2)}{2} - (d_{\beta_0} - rs)(x-s) \frac{(H^2)}{2} \right)}{xr - d_{\beta_0}} \\
&= \frac{(x-s) \left(a_{\beta_0} - d_{\beta_0}x \frac{(H^2)}{2} + s(rx - d_{\beta_0}) \frac{(H^2)}{2} \right)}{xr - d_{\beta_0}}.
\end{aligned}$$

Hence we define $\omega_{s,x}$ by

$$(4.8) \quad \frac{(\omega_{s,x}^2)}{2} = \frac{(x-s) \left(a_{\beta_0} - d_{\beta_0}x \frac{(H^2)}{2} + s(rx - d_{\beta_0}) \frac{(H^2)}{2} \right)}{xr - d_{\beta_0}}.$$

Corollary 4.2.5. *Assume that $\omega_{s,x}$ is general with respect to v . Then $\mathcal{M}_{(\beta, \omega_{s,x})}(v)$ does not depend on the choice of s .*

Proof. For the Mukai vector w_1 , we set $X_1 := M_H(w_1)$ and let \mathbf{E} be a universal family on $X \times X_1$. The isomorphism $\mathcal{M}_{(\beta, \omega_{s,x})}(v) \rightarrow \mathcal{M}_H^{\gamma'}(w)^{ss}$ is defined by the Fourier-Mukai transform $\Phi_{X \rightarrow X_1}^{\mathbf{E}[1]}$, which is independent of the choice of s . Since $\mathcal{M}_H^{\gamma'}(w)^{ss}$ is independent of the choice of s , we get the claim. \square

4.3. Ample line bundles on $M_{(\beta, \omega)}(v)$. We fix β in this subsection. We set

$$(4.9) \quad \varphi_\omega := \frac{r \frac{(\omega^2)}{2} - a_\beta}{d_\beta} = \frac{\operatorname{Re} Z_{(\beta, \omega)}(v)}{\operatorname{Im} Z_{(\beta, \omega)}(v)}(H, \omega).$$

Then

$$\{\varphi_\omega | \omega \in \mathbb{R}_{>0}H\} = \left(-\frac{a_\beta}{d_\beta}, \infty \right).$$

We set

$$\begin{aligned}
(4.10) \quad \xi_\omega &:= \frac{(\omega^2)}{2d_\beta} (r(H + (H, \beta)\varrho_X) + d_\beta(H^2)\varrho_X) - \frac{1}{d_\beta} (a_\beta(H + (H, \beta)\varrho_X) + d_\beta(H^2)e^\beta) \\
&= \varphi_\omega \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right) - (H^2) \left(e^\beta - \frac{a_\beta}{r} \varrho_X \right), \quad (r \neq 0).
\end{aligned}$$

For $\omega = \omega_\lambda$, $\lambda \in \mathbb{Q}$, we set $\gamma := \beta + \lambda H$. Let $w_\lambda = r_1 e^{\beta + \lambda H}$ be a primitive isotropic Mukai vector with $r_1 \lambda > 0$. We set $X_1 := M_H^\beta(w_\lambda)$.

Lemma 4.3.1. *By the Fourier-Mukai transform $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}^\alpha(X_1)$, we have*

$$(4.11) \quad \Phi(\xi_\omega) = \frac{1}{|r_1|(d_\beta - \lambda r)} (\operatorname{rk} w \hat{H} + (\hat{H}, c_1(w))\varrho_{X_1}),$$

where $w = \Phi(v)$. We also have

$$(4.12) \quad -\frac{\operatorname{rk} w}{r} \hat{\Phi} \left(e^{\gamma'} + \frac{\langle e^{\gamma'}, w \rangle}{\operatorname{rk} w} \varrho_{X_1} \right) = \left(e^\beta - \frac{a_\beta}{r} \varrho_X \right) + \lambda \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right).$$

Proof. We note that

$$\begin{aligned}
(4.13) \quad e^\gamma - \frac{a_\gamma}{r} \varrho_X &= e^\beta + \lambda(H + (H, \beta)\varrho_X) + \frac{(H^2)}{2} \lambda^2 \varrho_X - \frac{a_\gamma}{r} \varrho_X \\
&= e^\beta - \frac{a_\beta}{r} \varrho_X + \lambda \left(H + (H, \beta)\varrho_X + \frac{d_\beta}{r} (H^2)\varrho_X \right) \\
&= \left(e^\beta - \frac{a_\beta}{r} \varrho_X \right) + \lambda \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right)
\end{aligned}$$

and

$$(4.14) \quad \varphi_{\omega_\lambda} = \frac{a_\beta - r\lambda^2 \frac{(H^2)}{2}}{r\lambda - d_\beta} = \frac{a_\gamma + d_\gamma \lambda(H^2)}{-d_\gamma}.$$

Then we see that

$$\begin{aligned}
(4.15) \quad \xi_{\omega_\lambda} &= \frac{a_\gamma + d_\gamma \lambda (H^2)}{-d_\gamma} \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right) - \left(e^\gamma - \frac{a_\gamma}{r} \varrho_X \right) (H^2) + (H^2) \lambda \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right) \\
&= \frac{a_\gamma}{-d_\gamma} \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right) - \left(e^\gamma - \frac{a_\gamma}{r} \varrho_X \right) (H^2) \\
&= \frac{a_\gamma}{-d_\gamma} (H + (H, \gamma) \varrho_X) - (H^2) e^\gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
(4.16) \quad \Phi(\xi_{\omega_\lambda}) &= \frac{-r_1 a_\gamma}{|r_1| d_\gamma} (\hat{H} + (\hat{H}, \gamma') \varrho_{X_1}) + \frac{(H^2)}{r_1} \varrho_{X_1} \\
&= \frac{\text{rk } w}{|r_1| d_\gamma} (\hat{H} + (\hat{H}, \gamma') \varrho_{X_1}) + \frac{(H, d_\gamma H)}{r_1 d_\gamma} \varrho_{X_1} \\
&= \frac{1}{|r_1| d_\gamma} (\text{rk } w \hat{H} + (\hat{H}, c_1(w)) \varrho_{X_1}),
\end{aligned}$$

where we used the equality

$$c_1(w) = \frac{r_1}{|r_1|} d_\gamma \hat{H} + \text{rk } w \gamma' + D_{\gamma'}(w).$$

We also have

$$\begin{aligned}
(4.17) \quad -\frac{\text{rk } w}{r} \hat{\Phi} \left(e^{\gamma'} + \frac{\langle e^{\gamma'}, w \rangle}{\text{rk } w} \varrho_{X_1} \right) &= e^\gamma - \frac{a_\gamma}{r} \varrho_X \\
&= \left(e^\beta - \frac{a_\beta}{r} \varrho_X \right) + \lambda \left(H + \left(H, \frac{c_1(v)}{r} \right) \varrho_X \right).
\end{aligned}$$

□

Assume that $v_2 := r_2 e^\beta + a_2 \varrho_X + (d_2 H + D_2) + (d_2 H + D_2, \beta) \varrho_X$ satisfies

$$(4.18) \quad \frac{(\omega^2)}{2} = \frac{d_2 a_\beta - d_\beta a_2}{d_2 r - d_\beta r_2}.$$

Then

$$(4.19) \quad \varphi_\omega = \frac{r_2 a_\beta - r a_2}{d_2 r - d_\beta r_2}.$$

Hence

$$(4.20) \quad \langle v_2, \xi_\omega \rangle = \varphi_\omega \left(d_2 - \frac{r_2}{r} d_\beta \right) (H^2) - \left(\frac{a_\beta}{r} r_2 - a_2 \right) (H^2) = 0.$$

From now on, we assume that $\mathfrak{k} = \mathbb{C}$. Then we have a homomorphism

$$\theta_v : v^\perp \rightarrow H^2(M_{(\beta, \omega)}(v), \mathbb{Z})$$

which preserves the Hodge structures. If X is a $K3$ surface and v is a primitive Mukai vector with $\langle v^2 \rangle \geq 2$, then $M_{(\beta, \omega)}(v)$ is an irreducible symplectic manifold deformation equivalent to $\text{Hilb}_X^{\langle v^2 \rangle/2+1}$ by Theorem 4.1.1 and [11]. We regard $H^2(M_{(\beta, \omega)}(v), \mathbb{Z})$ as a lattice by the Beauville's bilinear form ([2]). Then θ_v is an isometry. If X is an abelian surface, then we have the albanese morphism $\mathfrak{a} : M_{(\beta, \omega)}(v) \rightarrow X \times \hat{X}$, which is an étale locally trivial fibration. Let $K_{(\beta, \omega)}(v)$ be the albanese fiber. Assume that v is primitive and $\langle v^2 \rangle \geq 6$. If ω is general, then $K_{(\beta, \omega)}(v)$ is an irreducible symplectic manifold which is deformation equivalent to the generalized Kummer variety constructed by Beauville [2]. We have an isomorphism

$$(4.21) \quad v^\perp \rightarrow H^2(M_{(\beta, \omega)}(v), \mathbb{Z}) \rightarrow H^2(K_{(\beta, \omega)}(v), \mathbb{Z})$$

which preserves the Hodge structures (cf. [11]). We also denote this map by θ_v . Thus we have an isomorphism

$$(4.22) \quad \theta'_v : v^\perp \cap H^*(X, \mathbb{Z})_{\text{alg}} \rightarrow \text{NS}(M_{(\beta, \omega)}(v)) \rightarrow \text{NS}(K_{(\beta, \omega)}(v))$$

as the restriction of θ_v .

Proposition 4.3.2. *Assume that ω belongs to a chamber I . Then $\theta_v(\xi_\omega) \in \text{NS}(M_{(\beta, \omega)}(v))_{\mathbb{R}}$ belongs to the ample cone of $M_{(\beta, \omega)}(v)$.*

Proof. For $\omega_\lambda \in I$, $\lambda \in \mathbb{Q}$, we take the isomorphism $\Phi : M_{(\beta, \omega_\lambda)}(v) \rightarrow M_{\hat{H}}^{\gamma'}(w)$. By Lemma 4.3.1 and Lemma 5.2.2,

$$(4.23) \quad \begin{aligned} \Phi(\theta_v(\xi_{\omega_\lambda})) &= \frac{\text{rk } w}{|r_1|(d_\beta - \lambda r)} \theta_w \left(\hat{H} + \left(\hat{H}, \frac{c_1(w)}{\text{rk } w} \right) \varrho_{X_1} \right) \\ &= \frac{\text{rk } w}{|r_1|(d_\beta - \lambda r)} \mathcal{L}(\xi_1) \end{aligned}$$

is a nef divisor on $M_{\hat{H}}^{\gamma'}(w)$. Hence $\theta_v(\xi_{\omega_\lambda})$ belongs to the nef cone of $M_{(\beta, \omega)}(v)$. Moreover by Lemma 5.2.2, $\Phi(\theta_v(\xi_\omega))$, $\omega \in I$ spans a 2-plane containing an ample divisor. Hence $\theta_v(\xi_\omega)$, $\omega \in I$ belongs to the ample cone of $M_{(\beta, \omega)}(v)$. \square

Corollary 4.3.3. *Let v be a primitive Mukai vector.*

- (1) *Assume that X is a K3 surface with $\text{NS}(X) = \mathbb{Z}H$ and $\langle v^2 \rangle \geq 2$. For a chamber $I = (\omega_1, \omega_2) \subset \mathbb{R}_{>0}H$ such that ω_1, ω_2 belong to walls,*

$$(4.24) \quad \text{Amp}(M_{(\beta, \omega)}(v))_{\mathbb{R}} \supset \theta_v(\{\mathbb{R}_{>0}\xi_\omega | \omega \in I\}).$$

- (2) *Assume that X is an abelian surface and $\langle v^2 \rangle \geq 6$. For a chamber $I = (\omega_1, \omega_2) \subset \mathbb{R}_{>0}H$ such that ω_1, ω_2 belong to walls,*

$$\text{Amp}(M_{(\beta, \omega)}(v))_{\mathbb{R}} \cap L = \theta_v(\{\mathbb{R}_{>0}\xi_\omega | \omega \in I\}),$$

where

$$L := \{\theta_v(x) | \langle x, v \rangle = 0, x \in \mathbb{R}e^\beta + \mathbb{R}(H + (H, \beta)\varrho_X) + \mathbb{R}\varrho_X\}.$$

In particular, if $\text{NS}(X) = \mathbb{Z}H$, then

$$\text{Amp}(K_{(\beta, \omega)}(v))_{\mathbb{R}} = \theta_v(\{\mathbb{R}_{>0}\xi_\omega | \omega \in I\}).$$

Proof. (1) is obvious by Proposition 4.3.2.

(2) Assume that ω belongs to a boundary of I defined by a wall W_{v_1} . We set $v_2 := v - v_1$. We may assume that $\phi_{(\beta, \omega)}(v_1) < \phi_{(\beta, \omega)}(v)$. Assume that $\langle v_1, v_2 \rangle \geq 2$ and there are $\sigma_{(\beta, \omega)}$ -stable objects E_i , $i = 1, 2$ with $v(E_i) = v_i$. Let P be the projective space associated to $\text{Ext}^1(E_2, E_1)$. We take the associated extension

$$0 \rightarrow E_1(\lambda) \rightarrow E \rightarrow E_2 \rightarrow 0$$

on $P \times X$, where $\mathcal{O}_P(\lambda)$ is the tautological line bundle. Then by using (4.20), we see that $\theta_v(\xi_\omega)|_P = \mathcal{O}_P$. Thus $\theta_v(\xi_\omega)$ is not ample.

We next treat the remaining case. By Proposition 5.3.2, replacing v_1 by another v_1 , we may assume that (i) $v = v_1 + nv_2$, $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 0$ and $\langle v_1, v_2 \rangle = 1$ or (ii) $v = v_1 + v_2 + v_3$, $\langle v_i^2 \rangle = 0$, $\langle v_i, v_j \rangle = 1$, ($i \neq j$). In the first case, we note that $n = \langle v^2 \rangle / 2 \geq 2$. We take $E_1 \in M_{(\beta, \omega)}(v_2)$ and $E_2 \in M_{(\beta, \omega)}(v - 2v_2)$. Since $\text{Ext}^1(E_1, E_1) \cong \mathbb{C}$, we have a family of non-trivial extensions F of E_1 by E_1 parametrized by a projective line P . Then $F \oplus E_2$ is a family of semi-stable objects and we have $\theta_v(\xi_\omega)|_P = \mathcal{O}_P$. Thus $\theta_v(\xi_\omega)$ is not ample. In the second case, for the \mathbb{P}^1 -bundles in Corollary 5.3.6, we can easily see that $\theta_v(\xi_\omega)|_{D_i}$ is trivial along the fibers of the \mathbb{P}^1 -bundles. Thus $\theta_v(\xi_\omega)$ is not ample. \square

Remark 4.3.4. The homomorphism

$$\theta'_v : v^\perp \cap H^*(X, \mathbb{Z})_{\text{alg}} \rightarrow \text{NS}(M_{(\beta, \omega)}(v))_{\mathbb{Q}}$$

is defined over any field \mathbb{k} . Replacing θ_v by θ'_v , Proposition 4.3.2 holds over any field \mathbb{k} .

5. APPENDIX

5.1. Another proof of Theorem 2.2.1. Assume that X_1 is a fine moduli scheme, that is, \mathbf{E} is an untwisted object. Then Theorem 2.2.1 directly follows from [4, Prop. 10.3], as we explain below.

We note that

$$(5.1) \quad \begin{aligned} e^{\beta + \sqrt{-1}\omega} &= e^\gamma e^{(\beta - \gamma) + \sqrt{-1}\omega} \\ &= e^\gamma + \left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta - \gamma, \omega) \right) \varrho_X + (\beta - \gamma + \sqrt{-1}\omega + (\beta - \gamma + \sqrt{-1}\omega, \gamma) \varrho_X). \end{aligned}$$

Hence

$$\begin{aligned}
(5.2) \quad \Phi(e^{\beta+\sqrt{-1}\omega}) &= -r_1 \left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta-\gamma, \omega) \right) e^{\gamma'} - \frac{1}{r_1} \varrho_{X_1} \\
&\quad + \frac{r_1}{|r_1|} \left(\widehat{\beta} - \widehat{\gamma} + \sqrt{-1}\widehat{\omega} + (\beta-\gamma + \sqrt{-1}\omega, \gamma) \varrho_{X_1} \right) \\
&= -r_1 \left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta-\gamma, \omega) \right) e^{\gamma' + \widehat{\xi} + \sqrt{-1}\widehat{\eta}},
\end{aligned}$$

where

$$\begin{aligned}
(5.3) \quad \xi &= -\frac{1}{|r_1|} \frac{1}{\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta-\gamma, \omega)^2} \left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} (\beta-\gamma) + (\beta-\gamma, \omega) \omega \right), \\
\eta &= -\frac{1}{|r_1|} \frac{1}{\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta-\gamma, \omega)^2} \left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \omega - (\beta-\gamma, \omega) (\beta-\gamma) \right).
\end{aligned}$$

Since $\langle \Phi(e^{\beta+\sqrt{-1}\omega}), x \rangle = \langle e^{\beta+\sqrt{-1}\omega}, \widehat{\Phi}(x) \rangle = Z_{(\beta, \omega)}(\widehat{\Phi}(x))$, we get the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{D}(X) & \xrightarrow{\Phi} & \mathbf{D}(X_1) \\
Z_{(\beta, \omega)} \downarrow & \circlearrowleft & \downarrow Z_{(\gamma' + \widehat{\xi}, \widehat{\eta})} \\
\mathbb{C} & \xrightarrow{\times \zeta^{-1}} & \mathbb{C}
\end{array}$$

with

$$\zeta = -r_1 \left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta-\gamma, \omega) \right).$$

We set $\beta = \gamma - \lambda H - \nu$ with $\nu \in H^\perp$. Then we see that

$$\begin{aligned}
(5.4) \quad & \frac{((\beta-\gamma)^2) - (\omega^2)}{2} \omega - (\beta-\gamma, \omega) (\beta-\gamma) \\
&= \frac{\lambda^2(H^2) + (\nu^2) - (\omega^2)}{2} \frac{(H, \omega)}{(H^2)} H - \lambda(H, \omega) (\lambda H + \nu) \\
&= (H, \omega) \left(\frac{-\lambda^2(H^2) + (\nu^2) - (\omega^2)}{2(H^2)} H - \lambda \nu \right) \\
&= - (H, \omega) L
\end{aligned}$$

where L is defined in Definition 2.1.2, and

$$\begin{aligned}
(5.5) \quad & \frac{((\beta-\gamma)^2) - (\omega^2)}{2} (\beta-\gamma) + (\beta-\gamma, \omega) \omega \\
&= -\frac{\lambda^2(H^2) + (\nu^2) - (\omega^2)}{2} (\lambda H + \nu) + \lambda(H, \omega) \frac{(H, \omega)}{(H^2)} H \\
&= -\frac{\lambda^2(H^2) + (\nu^2) - (\omega^2)}{2} (\lambda H + \nu) + \lambda(\omega^2) H \\
&= -\frac{\lambda^2(H^2) + (\nu^2) + (\omega^2)}{2} \lambda H - \frac{\lambda^2(H^2) + (\nu^2) - (\omega^2)}{2} \nu.
\end{aligned}$$

Proposition 5.1.1. [4, Prop. 10.3] *Assume that X_1 is a fine moduli space of $\sigma_{(\beta, \omega)}$ -stable objects, that is, \mathbf{E} is an untwisted object. Then $\widehat{\eta}$ is ample and Φ preserves the Bridgeland stability condition. Thus F is $\sigma_{(\beta, \omega)}$ -semi-stable if and only if $\Phi(F)$ is $\sigma_{(\gamma' + \widehat{\xi}, \widehat{\eta})}$ -semi-stable.*

If $\phi_{(\beta, \omega)}(F) = \phi_{(\beta, \omega)}(r_1 e^\gamma)$, then $\phi_{(\gamma' + \widehat{\xi}, \widehat{\eta})}(\Phi(F)) = \phi_{(\gamma' + \widehat{\xi}, \widehat{\eta})}(\mathbf{e}_{x_1}[-1]) \equiv 0 \pmod{2\mathbb{Z}}$. Hence F is a $\sigma_{(\beta, \omega)}$ -semi-stable object with $\phi_{(\beta, \omega)}(F) = \phi_{(\beta, \omega)}(r_1 e^\gamma)$ if and only if $\Phi(F)[1]$ is a $\sigma_{(\gamma' + \widehat{\xi}, \widehat{\eta})}$ -semi-stable object with $\phi_{(\gamma' + \widehat{\xi}, \widehat{\eta})}(\Phi(F)[1]) = 1$. This is equivalent to the condition that $H^{-1}(\Phi(F)[1])$ is a μ -semi-stable torsion free sheaf of $(c_1(H^{-1}(\Phi(F)[1])(-\widehat{\xi})), \widehat{\eta}) = 0$ with respect to $\widehat{\eta}$ and that $H^0(\Phi(F)[1])$ is a 0-dimensional sheaf. Thus we get another proof of Theorem 2.2.1.

Remark 5.1.2. For $E \in K(X_1)$, (2.7) is equivalent to $(c_1(E(-\gamma' - \widehat{\xi})), \widehat{L}) = 0$:

Indeed we first note that

$$\begin{aligned}
& r_1^2 \left\{ \left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2 \right\} \frac{(\xi, \eta)}{(H, \omega)} \\
&= \frac{\lambda(\lambda^2(H^2) + (\omega^2) + (\nu^2))(\lambda^2(H^2) + (\omega^2) - (\nu^2))}{4} + \lambda(\nu^2) \frac{\lambda^2(H^2) - (\omega^2) + (\nu^2)}{2} \\
(5.6) \quad &= \lambda \left\{ \left(\frac{\lambda^2(H^2) - (\omega^2) - (\nu^2)}{2} \right)^2 + \lambda^2(H^2)(\omega^2) \right\} \\
&= \lambda \left\{ \left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2 \right\}.
\end{aligned}$$

Since

$$L = \frac{\left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2}{(\omega, H)} |r_1| \eta,$$

we have

$$(5.7) \quad \lambda = r_1^2 \left\{ \left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2 \right\} \frac{(\xi, \eta)}{(\omega, H)} = |r_1|(L, \xi).$$

Thus we get

$$\begin{aligned}
(5.8) \quad & \frac{\left(\frac{((\beta - \gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2}{(\omega, H)} |r_1| \left((c_1(E(-\gamma')), \hat{\eta}) - \text{rk } E(\hat{\xi}, \hat{\eta}) \right) = (c_1(E(-\gamma')), \hat{L}) - \frac{\text{rk } E}{|r_1|} \lambda \\
&= (c_1(E(-\gamma' - \hat{\xi})), \hat{L}).
\end{aligned}$$

5.2. Polarizations on the moduli spaces of stable sheaves. Let X be a smooth projective surface with an ample divisor H . In this section, we study Simpson's polarization [10] of the moduli spaces of β -twisted stable sheaves. For a topological invariant v (e.g. Chern character or the equivalence class in the Grothendieck group $K(X)_{\text{top}}$ of topological vector bundles), $\overline{M}_H^\beta(v)$ denotes the moduli space of β -twisted semi-stable sheaves. We take a locally free sheaf G with $\frac{c_1(G)}{\text{rk } G} = \beta$. Let Q^{ss} be the open subscheme of $\text{Quot}_{G(-nH) \otimes V/X}$ such that $\overline{M}_H^\beta(v) = Q^{ss} // GL(V)$, where V is a vector space of dimension $\chi(G, E(nH))$, $E \in \overline{M}_H^\beta(v)$ and the action of $GL(V)$ is the natural one coming from the action on $G(-nH) \otimes V$. Let \mathcal{Q} be the universal quotient on $Q^{ss} \times X$. Then $\mathcal{Q}_{\{q\} \times X}$ is G -twisted semi-stable for all $q \in Q^{ss}$ and \mathcal{Q} is $GL(V)$ -linearized. By the construction of the moduli space, we have a $GL(V)$ -equivariant isomorphism $V \rightarrow p_{Q^{ss}*}(G^\vee \otimes \mathcal{Q}(nH))$. We set

$$\begin{aligned}
(5.9) \quad \mathcal{L}_{m,n} &:= \det p_{Q^{ss}*}(G^\vee \otimes \mathcal{Q}((n+m)H))^{\otimes P(n)} \otimes \det p_{Q^{ss}*}(G^\vee \otimes \mathcal{Q}(nH))^{\otimes (-P(m+n))} \\
&= \det p_{Q^{ss}*}(G^\vee \otimes \mathcal{Q}((n+m)H))^{\otimes P(n)} \otimes \det V^{\otimes (-P(m+n))},
\end{aligned}$$

where $P(n) := \chi(G, E(n))$ is the G -twisted Hilbert polynomial of $E \in \overline{M}_H^\beta(v)$. It is a $GL(V)$ -linearized line bundle on Q^{ss} , i.e., $\mathcal{L}_{m,n} \in \text{Pic}^{GL(V)}(Q^{ss})$. By the construction of the moduli space, we get the following.

Lemma 5.2.1. $\mathcal{L}_{m,n}$, $m \gg n \gg 0$ is the pull-back of a relatively ample line bundle on $\overline{M}_H^\beta(v)$.

We set

$$\begin{aligned}
(5.10) \quad \xi_1 &:= H + \left(H, \frac{c_1(v)}{r} - \frac{K_X}{2} \right) \varrho_X \\
\xi_2 &:= - \left(e^\beta - \frac{\chi(e^\beta, v)}{r} \varrho_X \right).
\end{aligned}$$

For $n_1, n_2 \in \mathbb{Q}$, let $\mathcal{L}(n_1\xi_1 + n_2\xi_2) \in \text{NS}(\overline{M}_H^\beta(v))_{\mathbb{Q}}$ be an algebraic equivalence class of a \mathbb{Q} -line bundle such that

$$q^*(\mathcal{L}(n_1\xi_1 + n_2\xi_2)^{\otimes N}) = \det p_{Q^{ss}*}(\mathcal{Q} \otimes F^\vee) \in \text{Pic}^{GL(V)}(Q^{ss}),$$

where $F \in \mathbf{D}(X)$ satisfies $\text{ch}(F) = N(n_1\xi_1 + n_2\xi_2)$, $N \gg 0$. Then we have a homomorphism

$$\begin{aligned}
(5.11) \quad \mathbb{Q}^{\oplus 2} &\rightarrow \text{NS}(\overline{M}_H^\beta(v))_{\mathbb{Q}} \\
(n_1, n_2) &\mapsto \mathcal{L}(n_1\xi_1 + n_2\xi_2).
\end{aligned}$$

Lemma 5.2.2. $\mathcal{L}(\xi_1 + \varepsilon\xi_2)$ is a \mathbb{Q} -ample divisor on $\overline{M}_H^\beta(v)$ for $0 < \varepsilon \ll 1$. In particular, $\mathcal{L}(\xi_1)$ defines a nef divisor.

Proof. We note that

$$\begin{aligned}
& \frac{\chi(G^\vee \otimes E(nH))}{\text{rk } G \text{ rk } E} \frac{\text{ch}(G^\vee((n+m)H))}{\text{rk } G} - \frac{\chi(G^\vee \otimes E((n+m)H))}{\text{rk } G \text{ rk } E} \frac{\text{ch}(G^\vee(nH))}{\text{rk } G} \\
&= \left(\frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E} + n \left(H, \frac{c_1(E)}{\text{rk } E} - \frac{c_1(G)}{\text{rk } G} - \frac{K_X}{2} \right) + \frac{(H^2)}{2} n^2 \right) \\
&\quad \times \left(\frac{\text{ch}(G^\vee)}{\text{rk } G} + (n+m) \left(H - \left(\frac{c_1(G)}{\text{rk } G}, H \right) \varrho_X \right) + \frac{(H^2)}{2} (n+m)^2 \varrho_X \right) \\
&\quad - \left(\frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E} + (n+m) \left(H, \frac{c_1(E)}{\text{rk } E} - \frac{c_1(G)}{\text{rk } G} - \frac{K_X}{2} \right) + \frac{(H^2)}{2} (n+m)^2 \right) \\
&\quad \times \left(\frac{\text{ch}(G^\vee)}{\text{rk } G} + n \left(H - \left(\frac{c_1(G)}{\text{rk } G}, H \right) \varrho_X \right) + \frac{(H^2)}{2} n^2 \varrho_X \right) \\
&= m \left(n(n+m) \frac{(H^2)}{2} - \frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E} \right) \left(-H + \left(H, \frac{c_1(E)}{\text{rk } E} - \frac{K_X}{2} \right) \varrho_X \right) \\
&\quad + m \left((2n+m) \frac{(H^2)}{2} + \left(H, \frac{c_1(E)}{\text{rk } E} - \frac{c_1(G)}{\text{rk } G} - \frac{K_X}{2} \right) \right) \left(\frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E} \varrho_X - \frac{\text{ch } G^\vee}{\text{rk } G} \right)
\end{aligned} \tag{5.12}$$

and

$$\frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E} \varrho_X - \frac{\text{ch } G^\vee}{\text{rk } G} = \xi_2^\vee.$$

Since

$$\lim_{m \rightarrow \infty} \frac{(2n+m) \frac{(H^2)}{2} + \left(H, \frac{c_1(E)}{\text{rk } E} - \frac{c_1(G)}{\text{rk } G} - \frac{K_X}{2} \right)}{n(n+m) \frac{(H^2)}{2} - \frac{\chi(G^\vee \otimes E)}{\text{rk } G \text{ rk } E}} = \frac{1}{n}$$

and n is an arbitrary large integer, we get the claim. \square

5.3. Some results to study the nef cone. In this subsection, we shall give some results which are used in the proof of Corollary 4.3.3 (2). So assume that X is an abelian surface.

We start with the following lemma.

Lemma 5.3.1. *Let v_1, v_2 be Mukai vectors such that $Z_{(\beta, \omega)}(v_2) \in \mathbb{R}Z_{(\beta, \omega)}(v_1)$.*

- (1) *If $d_\beta(v_1), d_\beta(v_2) > 0$, then $\langle v_1, v_2 \rangle \geq 0$ and the equality holds only if $\langle v_1^2 \rangle = 0$ and $v_2 \in \mathbb{Q}v_1$.*
- (2) *If $\langle v_1, v_2 \rangle > 0$ and $d_\beta(v_1) > 0$, then $d_\beta(v_2) > 0$.*
- (3) *Assume that $\langle v_1, v_2 \rangle = 1$, $d_\beta(v_1) > 0$ and $\langle v_1^2 \rangle \leq \langle v_2^2 \rangle$, then $\langle v_1^2 \rangle = 0$ and $v_2 = v'_2 + nv_1$, where v'_2 is an isotropic Mukai vector with $d_\beta(v'_2) > 0$.*

Proof. (1) is a consequence of the formula

$$\begin{aligned}
\frac{\langle v_1, v_2 \rangle}{d_\beta(v_1)d_\beta(v_2)} &= -\frac{1}{2} \left(\left(\frac{D_\beta(v_1)}{d_\beta(v_1)} - \frac{D_\beta(v_2)}{d_\beta(v_2)} \right)^2 \right) + \frac{\langle v_1^2 \rangle}{2d_\beta(v_1)^2} + \frac{\langle v_2^2 \rangle}{2d_\beta(v_2)^2} \\
&\quad + \left(\frac{r_\beta(v_1)}{d_\beta(v_1)} - \frac{r_\beta(v_2)}{d_\beta(v_2)} \right) \left(\frac{a_\beta(v_1)}{d_\beta(v_1)} - \frac{a_\beta(v_2)}{d_\beta(v_2)} \right)
\end{aligned}$$

for $d_\beta(v_1), d_\beta(v_2) > 0$ (see also [9, Lem. 3.1.1]) and the assumption $Z_{(\beta, \omega)}(v_2) \in \mathbb{R}Z_{(\beta, \omega)}(v_1)$.

(2) If $d_\beta(v_2) < 0$, then applying (1) to v_1 and $-v_2$, $\langle v_1, v_2 \rangle < 0$, which is a contradiction. Therefore $d_\beta(v_2) \geq 0$. By $Z_{(\beta, \omega)}(v_2) \in \mathbb{R}Z_{(\beta, \omega)}(v_1)$, $d_\beta(v_2) \neq 0$. Thus the claim holds.

(3) If $\langle v_1^2 \rangle \geq 2$, then $\langle v_1, v_2 \rangle \geq 3$ by [9, Lem. 4.2.4 (1)]. Hence $\langle v_1^2 \rangle = 0$. We set $v'_2 := v_2 - \frac{\langle v_2^2 \rangle}{2} v_1$. Since $\langle v_1, v'_2 \rangle = 1$ and $d_\beta(v_1) > 0$, (2) implies that $d_\beta(v'_2) > 0$. \square

Proposition 5.3.2. *Assume that $v = \sum_{i=1}^s n_i v_i$, where v_i are primitive Mukai vectors such that $v_i \neq v_j$ ($i \neq j$) and $\phi_{(\beta, \omega)}(v_i) = \phi_{(\beta, \omega)}(v)$.*

- (1) *If $s \geq 4$, then there are $\sigma_{(\beta, \omega)}$ -stable objects E_1, E_2 such that $v = v(E_1) + v(E_2)$ and $\langle v(E_1), v(E_2) \rangle \geq 2$.*
- (2) *Assume that $s = 3$. Then there are $\sigma_{(\beta, \omega)}$ -stable objects E_1, E_2 such that $v = v(E_1) + v(E_2)$ and $\langle v(E_1), v(E_2) \rangle \geq 2$ unless $\langle v_i^2 \rangle = 0$, $\langle v_i, v_j \rangle = 1$, $i \neq j$ and $n_1 = n_2 = n_3 = 1$.*
- (3) *Assume that $s = 2$. Then there are $\sigma_{(\beta, \omega)}$ -stable objects E_1, E_2 such that $v = v(E_1) + v(E_2)$ and $\langle v(E_1), v(E_2) \rangle \geq 2$ unless (a) there is an isotropic Mukai vector w_1 such that $\langle v, w_1 \rangle = 1$ and $\phi_{(\beta, \omega)}(w_1) = \phi_{(\beta, \omega)}(v)$ or (b) $n_1 = n_2 = 1$, $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 0$ and $\langle v_1, v_2 \rangle = 1$.*

Remark 5.3.3. In the case of (a) in (3), we have $w_1 = v_1$, $n_2 = 1$ or $w_1 = v_2$ and $n_1 = 1$. We also have $\langle v_1, v_2 \rangle = 1$.

Proof. (1) If $s \geq 4$, then Lemma 5.3.4 below implies that there is a $\sigma_{(\beta, \omega)}$ -stable object E_1 with $v(E_1) = v - v_1$. Let E_2 be a $\sigma_{(\beta, \omega)}$ -stable object with $v(E_2) = v_1$. Then the claim holds.

(2) We first assume that $(n_1, n_2, n_3) \neq (1, 1, 1)$. We may assume that $n_1 > 1$. Then there is a $\sigma_{(\beta, \omega)}$ -stable object E_1 with $v(E_1) = v - v_1$. Let E_2 be a $\sigma_{(\beta, \omega)}$ -stable object with $v(E_2) = v_1$. Then the claim holds.

We next assume that $n_1 = n_2 = n_3 = 1$. We shall classify the Mukai vectors for which the claim does not hold. We divide the argument into two cases.

(2-1) If $\langle v_1, v_2 \rangle > 1$, then Lemma 5.3.4 (2) below implies that there is a $\sigma_{(\beta, \omega)}$ -stable object E_1 with $v(E_1) = v_1 + v_2$. For a $\sigma_{(\beta, \omega)}$ -stable object E_2 with $v(E_2) = v_3$, $\langle v(E_1), v(E_2) \rangle \geq 2$. Thus the claim holds.

(2-2) We next assume that $\langle v_i, v_j \rangle = 1$ for $i \neq j$. If $n := \langle v_1^2 \rangle / 2 > 0$, then $\langle v_1, v_i \rangle = 1$ ($i = 2, 3$) implies that $\langle v_2^2 \rangle = \langle v_3^2 \rangle = 0$ by Lemma 5.3.1 (3). Since $v'_1 := v_1 - nv_2$ satisfies $\langle v_1'^2 \rangle = 0$ and $\langle v'_1, v_2 \rangle = 1$, we get $d_\beta(v'_1) > 0$ by Lemma 5.3.1 (2). Then $0 < \langle v'_1, v_3 \rangle = \langle v_1, v_3 \rangle - n \leq 0$. Therefore $\langle v_1^2 \rangle = 0$. In this case, $\langle v^2 \rangle = 6$.

(3) We may assume that $\langle v_1^2 \rangle \leq \langle v_2^2 \rangle$. (3-1) We first assume that $\langle v_1, v_2 \rangle \geq 2$ and divide the argument into two cases.

- (i) Assume that $\langle v_1^2 \rangle > 0$. If there is a primitive isotropic Mukai vector w_1 such that $\langle v_1, w_1 \rangle = 1$, then $v_1 = u_1 + nw_1$, $\langle u_1^2 \rangle = 0$. Then $v = n_1u_1 + n_1nw_1 + n_2v_2$ and u_1, w_1 and v_2 are different primitive vectors. By (2), we get the claim.

Otherwise, by Lemma 5.3.4, there are $\sigma_{(\beta, \omega)}$ -stable objects E_1, E_2 with $v(E_1) = v_1, v(E_2) = v_2$. Since $\langle v_1, v_2 \rangle \geq 3$, we get the claim.

- (ii) Assume that $\langle v_1^2 \rangle = 0$. By Lemma 5.3.4, if there is an integer i with $n_i \geq 2$, then there is a $\sigma_{(\beta, \omega)}$ -stable object E_1 with $v(E_1) = v - v_i$. For a $\sigma_{(\beta, \omega)}$ -stable object E_2 with $v(E_2) = v_i$, $\langle v(E_1), v(E_2) \rangle \geq 2$. Thus the claim holds.

If $n_1 = n_2 = 1$, then for $\sigma_{(\beta, \omega)}$ -stable objects E_i ($i = 1, 2$) with $v(E_i) = v_i$, the claim holds.

(3-2) Assume that $\langle v_1, v_2 \rangle = 1$. By Lemma 5.3.1 (3), $\langle v_1^2 \rangle = 0$. We set $v'_2 := v_2 - \frac{\langle v_2^2 \rangle}{2}v_1$ and set $n'_1 := n_1 + n_2 \frac{\langle v_2^2 \rangle}{2}$. Then $v = n'_1v_1 + n_2v'_2$, $\langle v_1, v'_2 \rangle = 1$ and $\langle v_2'^2 \rangle = 0$.

If $n'_1 = 1$ or $n_2 = 1$, then v_1 or v'_2 is an isotropic Mukai vector w_1 with $\langle v, w_1 \rangle = 1$. Hence we assume that $n'_1, n_2 \geq 2$.

If $n'_1 = 2$, then (i) $n_1 = 2$ and $\langle v_2^2 \rangle = 0$, or (ii) $n_1 = 1$ and $n_2\langle v_2^2 \rangle / 2 = 1$. Since $n_2 \geq 2$, (ii) does not occur. The case (i) corresponds to (b). If $n'_1 \geq 3$, then Lemma 5.3.4 implies that there is a $\sigma_{(\beta, \omega)}$ -stable object E_1 with $v(E_1) = (n'_1 - 1)v_1 + n_2v'_2$. For a $\sigma_{(\beta, \omega)}$ -stable object E_2 with $v(E_2) = v_1$, $\langle v(E_1), v(E_2) \rangle = n_2 \geq 2$. Therefore the claim holds. \square

Lemma 5.3.4. *Let v be a Mukai vector such that $\langle v^2 \rangle > 0$ and $d_\beta(v) > 0$.*

- (1) *If there is no isotropic Mukai vector w_1 with $\langle v, w_1 \rangle = 1$ and $\phi_{(\beta, \omega)}(w_1) = \phi_{(\beta, \omega)}(v)$, then there is a $\sigma_{(\beta, \omega)}$ -stable object E with $v = v(E)$.*
- (2) *Assume that v has a decomposition $v = \sum_{i=1}^s n_i v_i$, where v_i are primitive Mukai vectors such that $v_i \neq v_j$ ($i \neq j$), $\langle v_i^2 \rangle \geq 0$ and $\phi_{(\beta, \omega)}(v_i) = \phi_{(\beta, \omega)}(v)$. Then there is a $\sigma_{(\beta, \omega)}$ -stable object E with $v = v(E)$, unless there is an isotropic Mukai vector w_1 with $\phi_{(\beta, \omega)}(w_1) = \phi_{(\beta, \omega)}(v)$ and $\langle v, w_1 \rangle = 1$. In particular if $s \geq 3$ or $s = 2$ and $\langle v_1, v_2 \rangle \geq 2$, then there is a $\sigma_{(\beta, \omega)}$ -stable object E with $v = v(E)$.*

Proof. (1) By [9, Prop. 4.2.5], $\mathcal{M}_{(\beta, \omega_+)}(v) \cap \mathcal{M}_{(\beta, \omega_-)}(v) \neq \emptyset$, if there is no isotropic Mukai vector w_1 with $\langle v, w_1 \rangle = 1$. Since $\langle v^2 \rangle > 0$ and $\mathcal{M}_{(\beta, \omega_+)}(v)$ is isomorphic to a moduli stack of semi-stable sheaves (Theorem 0.0.1), [7, Lem. 3.2] implies that there is a $\sigma_{(\beta, \omega_+)}$ -stable object E in $\mathcal{M}_{(\beta, \omega_+)}(v) \cap \mathcal{M}_{(\beta, \omega_-)}(v)$. Then E is $\sigma_{(\beta, \omega)}$ -stable. Indeed for a subobject E_1 of E with $\phi_{(\beta, \omega)}(E_1) = \phi_{(\beta, \omega)}(E)$, $\phi_{(\beta, \omega_\pm)}(E_1) \leq \phi_{(\beta, \omega_\pm)}(E)$. Then $\phi_{(\beta, \omega_\pm)}(E_1) = \phi_{(\beta, \omega_\pm)}(E)$, which implies that E is properly $\sigma_{(\beta, \omega_+)}$ -semi-stable. Therefore E is $\sigma_{(\beta, \omega)}$ -stable.

(2) We first assume that $s \geq 3$. Then for any isotropic Mukai vector w_1 with $\phi_{(\beta, \omega)}(w) = \phi_{(\beta, \omega)}(v)$, $\langle v, w_1 \rangle \geq 2$ since there are two different indices $1 \leq i, j \leq s$ such that

$$\langle v, w \rangle \geq n_i \langle v_i, w \rangle + n_j \langle v_j, w \rangle \geq n_i + n_j \geq 2.$$

Here we used Lemma 5.3.1 (1). By (1), the claim holds.

We next assume that $s = 2$. Let w_1 be an isotropic Mukai vector with $\phi_{(\beta, \omega)}(w_1) = \phi_{(\beta, \omega)}(v)$. We have $\langle w_1, v \rangle \geq n_1 + n_2 \geq 2$, if $w \neq v_1, v_2$. Hence if $\langle v_1, v_2 \rangle \geq 2$, then the assumption of (1) also holds. So the conclusion follows from (1).

If $\langle v_1, v_2 \rangle = 1$, then we may assume that $0 = \langle v_1^2 \rangle \leq \langle v_2^2 \rangle$ by Lemma 5.3.1 (3). In this case, $v = n'_1v_1 + n_2v'_2$, where $v'_2 = v_2 - \frac{\langle v_2^2 \rangle}{2}v_1$ and $n'_1 = n_1 + n_2 \frac{\langle v_2^2 \rangle}{2}$. If $n'_1, n_2 \geq 2$, then the assumption of (1) holds. So the conclusion follows from (1). If $n'_1 = 1$ or $n_2 = 1$, then there is an isotropic Mukai vector w_1 with $\langle v, w_1 \rangle = 1$. \square

Lemma 5.3.5. *Let v_i ($i = 1, 2, 3$) be primitive isotropic Mukai vectors such that $\langle v_i, v_j \rangle = 1$ ($i \neq j$), $\phi_{(\beta, \omega)}(v_i) = v$ and $v = \sum_{i=1}^3 v_i$. Let (β', ω') be a general pair which is close to (β, ω) .*

- (1) Assume that $\phi_{(\beta', \omega')}(v_1), \phi_{(\beta', \omega')}(v_2) < \phi_{(\beta', \omega')}(v)$. For $E_i \in \mathcal{M}_{(\beta, \omega)}(v_i)$, we take non-trivial extensions

$$0 \rightarrow E_1 \rightarrow E_{13} \rightarrow E_3 \rightarrow 0$$

and

$$0 \rightarrow E_2 \rightarrow E_{23} \rightarrow E_3 \rightarrow 0.$$

Then E_{13} and E_{23} are $\sigma_{(\beta', \omega')}$ -stable objects. We take non-trivial extensions

$$0 \rightarrow E_1 \rightarrow E_{123} \rightarrow E_{23} \rightarrow 0$$

and

$$0 \rightarrow E_2 \rightarrow E_{213} \rightarrow E_{13} \rightarrow 0.$$

Then E_{123} and E_{213} are $\sigma_{(\beta', \omega')}$ -stable objects.

- (2) Assume that $\phi_{(\beta', \omega')}(v_1), \phi_{(\beta', \omega')}(v_2) > \phi_{(\beta', \omega')}(v)$. For $E_i \in \mathcal{M}_{(\beta, \omega)}(v_i)$, we take non-trivial extensions

$$0 \rightarrow E_3 \rightarrow E_{31} \rightarrow E_1 \rightarrow 0$$

and

$$0 \rightarrow E_3 \rightarrow E_{32} \rightarrow E_2 \rightarrow 0.$$

Then E_{31} and E_{32} are $\sigma_{(\beta', \omega')}$ -stable objects. We take non-trivial extensions

$$0 \rightarrow E_{31} \rightarrow E_{312} \rightarrow E_2 \rightarrow 0$$

and

$$0 \rightarrow E_{32} \rightarrow E_{321} \rightarrow E_1 \rightarrow 0.$$

Then E_{312} and E_{321} are $\sigma_{(\beta', \omega')}$ -stable objects.

Proof. We only prove the $\sigma_{(\beta', \omega')}$ -stability of E_{123} in (1). Assume that E_{123} contains an object F with $\phi_{(\beta', \omega')}(F) > \phi_{(\beta', \omega')}(v)$. We may assume that $\phi_{(\beta, \omega)}(F) = \phi_{(\beta, \omega)}(v)$. Then we see that (i) $F \cong E_3$ or (ii) $E_{123}/F \cong E_1$ or (iii) $E_{123}/F \cong E_2$. If $F \cong E_3$, then $E_{23} \cong E_2 \oplus E_3$. If $E_{123}/F \cong E_1$, then $E_{123} \cong E_{23} \oplus E_1$. If $E_{123}/F \cong E_2$, then $E_1 \subset F$ and E_{23} contains $F/E_2 \cong E_3$. Therefore each case does not occur. Hence E_{123} is $\sigma_{(\beta', \omega')}$ -stable. \square

Since $\langle v_1 + v_2, v_3 \rangle = 2$, we get the following result.

Corollary 5.3.6. *Under the same assumptions of Lemma 5.3.5, $M_{(\beta', \omega')}(v)$ contains two \mathbb{P}^1 -bundles D_1, D_2 over $\prod_{i=1}^3 M_{(\beta, \omega)}(v_i)$.*

Remark 5.3.7. It seems that they intersect transversely. Thus D_1 and D_2 give an A_2 -type configuration.

Acknowledgement. The third author would like to thank Professor A. Maciocia for valuable discussions on Bridgeland's stability during the program "Moduli spaces" at the Newton Institute.

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